

# HIGHER DEPTH QUANTUM MODULAR FORMS, MULTIPLE EICHLER INTEGRALS, AND $\mathfrak{sl}_3$ FALSE THETA FUNCTIONS

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**ABSTRACT.** We introduce and study higher depth quantum modular forms. We construct two families of examples coming from rank two false theta functions, whose “companions” in the lower half-plane can be also realized both as double Eichler integrals and as non-holomorphic theta series having values of “double error” functions as coefficients. In particular, we prove that the false theta of  $\mathfrak{sl}_3$ , appearing in the character of the vertex algebra  $W^0(p)_{A_2}$ , can be written as the sum of two depth two quantum modular forms of positive integral weight.

## 1. INTRODUCTION AND STATEMENT OF RESULTS

In this paper, we study higher depth quantum modular forms which occur as rank two false theta functions coming from characters of the vertex algebra  $W^0(p)_{A_2}$ ,  $p \geq 3$ . Via asymptotic expansion we relate these to double Eichler integrals which may be viewed as purely non-holomorphic parts of indefinite theta functions.

Let us first recall the classical rank one case. Note that the derivative of a modular form is typically not a modular form anymore (only a so-called quasi-modular form). However, thanks to Bol’s identity, differentiating a weight  $2 - k \in -\mathbb{N}$  modular form  $k - 1$  times returns a modular form of weight  $k$ . Thus it is natural to consider holomorphic Eichler integrals. That is, if  $f(\tau) = \sum_{m \geq 1} c_f(m) q^m$  ( $q := e^{2\pi i \tau}$  with  $\tau \in \mathbb{H}$  throughout) is a modular form of weight  $k$ , then set

$$\tilde{f}(\tau) := \sum_{m \geq 1} \frac{c_f(m)}{m^{k-1}} q^m. \quad (1.1)$$

It easily follows, by Bol’s identity and the modularity of  $f$ , that the following function gets annihilated by differentiating  $k - 1$  times

$$R_f(\tau) := \tilde{f}(\tau) - \tau^{k-2} \tilde{f}\left(-\frac{1}{\tau}\right). \quad (1.2)$$

This yields that  $R_f$  is a polynomial of degree  $k - 2$  (the so called *period polynomial* of  $f$ ). So in particular  $R_f$  is much simpler than the starting function  $\tilde{f}$ . Note that  $\tilde{f}$  may also be written as an integral, namely, up to constants, it equals

$$\int_{\tau}^{i\infty} f(w)(w - \tau)^{k-2} dw. \quad (1.3)$$

Similarly  $R_f$  has an integral representation, namely up to constants it equals

$$\int_0^{i\infty} f(w)(w - \tau)^{k-2} dw.$$

A similar construction works for *weakly holomorphic modular forms*, i.e., those meromorphic modular forms which may only grow as  $v := \text{Im}(\tau) \rightarrow \infty$ . In this situation, (1.3) needs to be regularized. Moreover, note that there is a “companion integral” (again regularized)

$$I_g(\tau) := \int_{-\bar{\tau}}^{i\infty} g(w)(w + \tau)^{k-2} dw \quad (1.4)$$

for some weakly holomorphic modular form  $g$  in the sense that the corresponding period polynomial, defined analogously to (1.2), basically agrees with  $R_f$ .

In contrast, for half-integral weight modular forms there is no half-derivative and thus Bol’s identity does not apply. However, one can formally define the analogue of (1.1). This was first investigated by Zagier [25, 26] in connection to Kontsevich’s “strange function”

$$K(q) := \sum_{m \geq 0} (q; q)_m,$$

where for  $m \in \mathbb{N}_0 \cup \{\infty\}$ ,  $(a; q)_m := \prod_{j=0}^{m-1} (1 - aq^j)$  denotes the usual  $q$ -Pochhammer symbol. This function does not converge on any open subset of  $\mathbb{C}$ , but converges as a finite sum for  $q$  any root of unity. Zagier’s study of  $K$  depends on the identity

$$\sum_{m \geq 0} \left( \eta(\tau) - q^{\frac{1}{24}} (q; q)_m \right) = \eta(\tau) D(\tau) + \frac{1}{2} \tilde{\eta}(\tau), \quad (1.5)$$

with  $\eta(\tau) := q^{\frac{1}{24}} (q; q)_\infty = \sum_{m \geq 1} \left(\frac{12}{m}\right) q^{\frac{m^2}{24}}$ ,  $D(\tau) := -\frac{1}{2} + \sum_{m \geq 1} \frac{q^m}{1 - q^m}$  and  $\tilde{\eta}(\tau) := \sum_{m \geq 1} \left(\frac{12}{m}\right) m q^{\frac{m^2}{24}}$ . The key observation of Zagier is that in (1.5), the functions  $\eta(\tau)$  and  $\eta(\tau) D(\tau)$  vanish of infinite order as  $\tau \rightarrow \frac{h}{k} \in \mathbb{Q}$ . So at a root of unity  $\zeta$ ,  $K(\zeta)$  is essentially the limiting value of the Eichler integral of  $\eta$ , which he showed has quantum modular properties. Roughly speaking, Zagier defined “quantum modular forms” to be functions  $f : \mathcal{Q} \rightarrow \mathbb{C}$  ( $\mathcal{Q} \subseteq \mathbb{Q}$ ), such that the error of modularity ( $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z})$ )

$$f(\tau) - (c\tau + d)^{-k} f(M\tau) \quad (1.6)$$

is “nice”. The definition is intentionally vague to include many examples; in this paper we in particular require (1.6) to be real-analytic. For example,  $\tilde{f}$  is a quantum modular form, since  $R_f$  is a polynomial and thus, in particular, real-analytic. Additional examples appear in the study of limits of quantum invariants of 3-manifolds and knots [26], Kashaev invariants of  $(p, q)$ -torus knots [13, 14], and partial theta functions [10].

Motivated in part by vertex operator algebra theory, in [4] and [8], further (but similar) examples of quantum modular forms were investigated in the setup of characters of vertex algebra modules. These examples are given by characters of  $M_{r,s}$ , the atypical irreducible modules of the  $(1, p)$ -singlet algebra for  $p \geq 2$  [4, 6]. For  $r = 1$  and  $1 \leq s \leq p - 1$ , they take the particularly nice shape

$$\text{ch}_{M_{1,s}}(\tau) = \frac{F_{p-s,p}(p\tau)}{\eta(\tau)},$$

where

$$F_{j,p}(\tau) := \sum_{m \in \mathbb{Z}} \text{sgn} \left( m + \frac{j}{2p} \right) q^{\left( m + \frac{j}{2p} \right)^2}.$$

is a *false theta function*. It is called “false theta” since getting rid of the  $\text{sgn}$ -factor yields the theta function  $\sum_{m \in \mathbb{Z}} q^{(m + \frac{j}{2p})^2}$ , a modular form of weight  $\frac{1}{2}$ . Quantum modularity of  $F_{j,p}$  is now given by relating it to a non-holomorphic Eichler integral, as in (1.4). To be more precise, set

$$F_{j,p}^*(\tau) := \sqrt{2} \int_{-\bar{\tau}}^{i\infty} \frac{f_{j,p}(w)}{(-i(w + \tau))^{\frac{1}{2}}} dw,$$

where  $f_{j,p}$  is the cuspidal weight  $\frac{3}{2}$  theta function

$$f_{j,p}(\tau) := \sum_{m \in \mathbb{Z}} \left( m + \frac{j}{2p} \right) q^{\left( m + \frac{j}{2p} \right)^2}.$$

One can show that  $F_{j,p}(\tau)$  agrees for  $\tau = \frac{h}{k}$  with  $F_{j,p}^*(\tau)$  up to infinite order [4]. Quantum modularity then follows by the (mock) modular transformation of  $F_{j,p}^*$  which we recall in Lemma 2.3 below. By “mock-modular”, we mean that the extra term  $r_{f, \frac{d}{c}}$  in Lemma 2.3 prevents the function from being modular. However, there exists a “modular completion” in the sense that after multiplying it with a theta function,  $F_{j,p}^*$  is the “purely non-holomorphic part” of a non-holomorphic theta function corresponding to an indefinite quadratic form (of signature  $(1, 1)$ ). Its modularity now can be proven by using results of Zwegers [27, Section 2.2]. The functions  $F_{j,p}(p\tau)$ , especially if  $p = 2$ , have appeared in several studies of vertex algebras from different standpoints [3, 6, 11, 15].

In this paper we investigate higher-dimensional analogues. For this we consider certain  $q$ -series appearing in representation theory of vertex algebras and  $W$ -algebras. They are sometimes called *higher rank false theta functions* and are thoroughly studied in [4, 7]. They appear from extracting the constant term of certain multivariable Jacobi forms [4]. The constant term can be interpreted as the character of the zero weight space of the corresponding Lie algebra representation. In the case of the simple Lie algebra  $\mathfrak{sl}_3$ , the false theta function takes the following shape ( $p \in \mathbb{N}$ ,  $p \geq 2$ ):

$$F(q) := \sum_{\substack{m_1, m_2 \geq 1 \\ m_1 \equiv m_2 \pmod{3}}} \min(m_1, m_2) q^{p(m_1^2 + m_2^2 + m_1 m_2) - m_1 - m_2 + \frac{1}{p}} (1 - q^{m_1}) (1 - q^{m_2}) (1 - q^{m_1 + m_2}). \quad (1.7)$$

Below we decompose this function as  $F(q) = F_1(q) + F_2(q)$  with  $F_1$  and  $F_2$  defined in (3.1) and (3.2), respectively. The function  $F_1$  and  $F_2$  turn out to have generalized quantum modular properties. This connection goes via an analogue of (1.1). For instance, we show that  $F_1$  asymptotically agrees with an integral of the shape

$$\int_{-\bar{\tau}}^{i\infty} \int_{w_1}^{i\infty} \frac{f(w_1, w_2)}{\sqrt{-i(w_1 + \tau)} \sqrt{-i(w_2 + \tau)}} dw_1 dw_2$$

where  $f \in S_{\frac{3}{2}}(\chi_1, \Gamma) \otimes S_{\frac{3}{2}}(\chi_2, \Gamma)$  ( $\chi_j$  multipliers and  $\Gamma \subset \text{SL}_2(\mathbb{Z})$ ). Modular properties follow from the modularity of  $f$  which in turn gives quantum modular properties of  $F_1$ . The idea is that here the error of modularity (1.6) is less complicated than the original function. We call the resulting functions higher depth quantum modular forms (see Definition 3 for a precise definition). Roughly speaking (see Definition 3 for a precise definition), depth two quantum modular forms satisfy, in the simplest case, the modular transformation property  $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z})$

$$f(\tau) - (c\tau + d)^{-k} f(M\tau) \in \mathcal{Q}_\kappa(\Gamma) \mathcal{O}(R) + \mathcal{O}(R),$$

where  $\mathcal{Q}_\kappa(\Gamma)$  is the space of quantum modular forms of weight  $\kappa$  and  $\mathcal{O}(R)$  the space of real analytic functions on  $R \subset \mathbb{R}$ . Clearly, we can construct examples of depth two simply by multiplying two (depth one) quantum modular forms. Non-trivial examples arise from  $F$  (see Theorem 1.1 for precise statement).

**Theorem 1.1.** *For  $p \geq 3$ , the higher rank false theta function  $F$  can be written as the sum of two depth two quantum modular forms of weight one and two.*

We next turn to the modular completion of these Eichler integrals (see Proposition 8.1 for a more precise version). For theta functions associated to indefinite quadratic forms, the reader is referred to [1, 16, 19, 22].

**Theorem 1.2.** *There exists an indefinite theta function, defined via (8.1), of signature  $(2, 2)$  with “purely non-holomorphic” part  $\Theta(\tau)\mathcal{E}_1(\tau)$  where  $\Theta$  is a theta function of signature  $(2, 0)$ .*

The paper is organized as follows. In Section 2, we review basic results on special functions, non-holomorphic Eichler integrals, and “double error” functions  $M_2$  and  $E_2$ . We also recall the notion of quantum modular forms and introduce higher depth quantum modular forms. In Section 3, the  $\mathfrak{sl}_3$  higher rank false theta function  $F(q) = F_1(q) + F_2(q)$  is introduced. In Section 4, we determine the asymptotic behavior of  $F_1$  and  $F_2$  at roots of unity. In Section 5, we introduce multiple Eichler integrals and prove modular transformation formula for the double Eichler integrals. We also study certain linear combinations of double Eichler integrals associated to  $F_j$ . In Section 6, we express special double Eichler integrals as pieces of indefinite theta series. Based on results in this section, in Section 7, we prove the main result, Theorem 1.1, on quantum modularity of  $F$ . Section 8 deals with the completion of certain indefinite theta functions of signature  $(2, 2)$  associated to the companions of  $F_j$  proving Theorem 1.2. We conclude in Section 9 with several questions.

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## 2. PRELIMINARIES

**2.1. Special functions.** Define for  $u \in \mathbb{R}$ , which is essentially the error function.

$$E(u) := 2 \int_0^u e^{-\pi w^2} dw.$$

Its derivative is  $E'(u) = 2e^{-\pi u^2}$ . We furthermore have the representation

$$E(u) = \operatorname{sgn}(u) \left( 1 - \frac{1}{\sqrt{\pi}} \Gamma \left( \frac{1}{2}, \pi u^2 \right) \right), \quad (2.1)$$

where  $\Gamma(\alpha, u) := \int_u^\infty e^{-w} w^{\alpha-1} dw$  is the *incomplete gamma function* and where for  $u \in \mathbb{R}$ , we set

$$\operatorname{sgn}(u) := \begin{cases} 1 & \text{if } u > 0, \\ -1 & \text{if } u < 0, \\ 0 & \text{if } u = 0. \end{cases}$$

We also require the functional equation of the incomplete  $\Gamma$ -function

$$\Gamma\left(\frac{1}{2}, u\right) = -\frac{1}{2}\Gamma\left(-\frac{1}{2}, u\right) + \frac{1}{\sqrt{u}}e^{-u}. \quad (2.2)$$

Moreover, for  $u \neq 0$ , set

$$M(u) := \frac{i}{\pi} \int_{\mathbb{R}-iu} \frac{e^{-\pi w^2 - 2\pi i u w}}{w} dw.$$

We have

$$M(u) = E(u) - \operatorname{sgn}(u).$$

Thus, by (2.1)

$$M(u) = -\frac{\operatorname{sgn}(u)}{\sqrt{\pi}}\Gamma\left(\frac{1}{2}, \pi u^2\right). \quad (2.3)$$

This implies that the following bound holds

$$|M(u)| \leq 2e^{-\pi u^2}.$$

We next turn to two-dimensional analogues, following [1] (using slightly different notation).

Define  $E_2 : \mathbb{R} \times \mathbb{R}^2 \rightarrow \mathbb{R}$  by (throughout we denote the components of vectors just with subscripts)

$$E_2(\kappa; u) := \int_{\mathbb{R}^2} \operatorname{sgn}(w_1) \operatorname{sgn}(w_2 + \kappa w_1) e^{-\pi((w_1 - u_1)^2 + (w_2 - u_2)^2)} dw_1 dw_2.$$

Note that

$$E_2(\kappa; -u) = E_2(\kappa; u).$$

Moreover, also following [1], for  $u_2, u_1 - \kappa u_2 \neq 0$ :

$$M_2(\kappa; u_1, u_2) := -\frac{1}{\pi^2} \int_{\mathbb{R}-iu_2} \int_{\mathbb{R}-iu_1} \frac{e^{-\pi w_1^2 - \pi w_2^2 - 2\pi i(u_1 w_1 + u_2 w_2)}}{w_2(w_1 - \kappa w_2)} dw_1 dw_2$$

Then we have

$$\begin{aligned} M_2(\kappa; u_1, u_2) &= E_2(\kappa; u_1, u_2) - \operatorname{sgn}(u_2) M(u_1) \\ &\quad - \operatorname{sgn}(u_1 - \kappa u_2) M_1\left(\frac{u_2 + \kappa u_1}{\sqrt{1 + \kappa^2}}\right) - \operatorname{sgn}(u_1) \operatorname{sgn}(u_2 + \kappa u_1). \end{aligned} \quad (2.4)$$

Note that (2.4) extends the definition of  $M_2$  to  $u_2 = 0$  or  $u_1 = \kappa u_2$ . With  $x_1 := u_1 - \kappa u_2$ ,  $x_2 := u_2$ , a direct calculation shows that

$$\begin{aligned} M_2(\kappa; u_1, u_2) &= E_2(\kappa; x_1 + \kappa x_2, x_2) + \operatorname{sgn}(x_1) \operatorname{sgn}(x_2) \\ &\quad - \operatorname{sgn}(x_2) E(x_1 + \kappa x_2) - \operatorname{sgn}(x_1) E\left(\frac{\kappa x_1}{\sqrt{1 + \kappa^2}} + \sqrt{1 + \kappa^2} x_2\right). \end{aligned}$$

We have the first partial derivatives

$$M_2^{(0,1)}(\kappa; u_1, u_2) = \frac{2}{\sqrt{1 + \kappa^2}} e^{-\frac{\pi(u_2 + \kappa u_1)^2}{1 + \kappa^2}} M\left(\frac{u_1 - \kappa u_2}{\sqrt{1 + \kappa^2}}\right), \quad (2.5)$$

$$M_2^{(1,0)}(\kappa; u_1, u_2) = 2e^{-\pi u_1^2} M(u_2) + \frac{2\kappa}{\sqrt{1 + \kappa^2}} e^{-\frac{\pi(u_2 + \kappa u_1)^2}{1 + \kappa^2}} M\left(\frac{u_1 - \kappa u_2}{\sqrt{1 + \kappa^2}}\right), \quad (2.6)$$

and the limiting behavior (cf. [1, Proposition 3.3, iii])

$$M_2(\kappa; \lambda u_1, \lambda u_2) \sim -\frac{e^{-\pi\lambda^2(u_1^2+u_2^2)}}{\lambda^2\pi^2 u_2(u_1 - \kappa u_2)} \quad (\text{as } \lambda \rightarrow \infty). \quad (2.7)$$

We further need the statements to prove the modularity of an appearing indefinite theta function.

**Lemma 2.1.** *For  $u_3, u_4 + \kappa u_3 \neq 0$ , we have the following limits*

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0^+} E_2(\varepsilon\kappa; u_1, \varepsilon u_2 + \varepsilon^{-1}u_3) &= \operatorname{sgn}(u_3)E(u_1), \\ \lim_{\varepsilon \rightarrow 0^+} E_2(\kappa; \varepsilon u_1 + \varepsilon^{-1}u_3, \varepsilon u_2 + \varepsilon^{-1}u_4) &= \operatorname{sgn}(u_3)\operatorname{sgn}(u_4 + \kappa u_3). \end{aligned}$$

**2.2. Euler-Maclaurin summation formula.** We now state a special case of the Euler-Maclaurin summation formula as needed for this paper. We only give it in the two-dimensional case; the one-dimensional one can be concluded from it by viewing the second variable as constant.

Let  $B_m(x)$  be the  $n$ th Bernoulli polynomial defined by  $\frac{te^{xt}}{e^t-1} =: \sum_{m \geq 0} B_m(x) \frac{t^m}{m!}$ . We also require

$$B_m(1-x) = (-1)^m B_m(x).$$

The Euler-Maclaurin summation formula implies that, for  $\alpha \in (\mathbb{R}^+)^2$ ,  $F : \mathbb{R}^2 \rightarrow \mathbb{R}$  a  $C^\infty$ -function of rapid decay, we have (generalizing a result of [24])

$$\begin{aligned} \sum_{n \in \alpha + \mathbb{N}_0^2} F((n+\alpha)t) &\sim \frac{\mathcal{I}_F}{t^2} - \sum_{n_2 \geq 0} \frac{B_{n_2+1}(\alpha_2)}{(n_2+1)!} \int_0^\infty F^{(0, n_2)}(x_1, 0) dx_1 t^{n_2-1} \\ &- \sum_{n_1 \geq 0} \frac{B_{n_1+1}(\alpha_1)}{(n_1+1)!} \int_0^\infty F^{(n_1, 0)}(0, x_2) dx_2 t^{n_1-1} + \sum_{n_1, n_2 \geq 0} \frac{B_{n_1+1}(\alpha_1)}{(n_1+1)!} \frac{B_{n_2+1}(\alpha_2)}{(n_2+1)!} F^{(n_1, n_2)}(0, 0) t^{n_1+n_2}, \end{aligned} \quad (2.8)$$

where  $\mathcal{I}_F := \int_0^\infty \int_0^\infty F(x_1, x_2) dx_1 dx_2$ . Here by  $\sim$  we mean that the difference between the left- and the right-hand side is  $O(t^N)$  for any  $N \in \mathbb{N}$ .

**2.3. Shimura's theta functions.** We require transformation laws of certain theta functions studied by Shimura [20]. For  $\nu \in \{0, 1\}$ ,  $h \in \mathbb{Z}$ ,  $N, A \in \mathbb{N}$ , with  $A|N$ ,  $N|hA$ , define

$$\Theta_\nu(A, h, N; \tau) := \sum_{\substack{m \in \mathbb{Z} \\ m \equiv h \pmod{N}}} m^\nu q^{\frac{Am^2}{2N^2}}. \quad (2.9)$$

Recall the following transformation

$$\Theta_\nu(A, h, N; M\tau) = e\left(\frac{abAh^2}{2N^2}\right) \left(\frac{2Ac}{d}\right) \varepsilon_d^{-(\nu+1)} (c\tau + d)^{\frac{1}{2}+\nu} \Theta_\nu(A, ah, N; \tau) \quad (2.10)$$

for  $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(2N)$  with  $2|b$ . Here  $e(x) := e^{2\pi i x}$ , for odd  $d$ ,  $\varepsilon_d = 1$  or  $i$ , depending on whether  $d \equiv 1 \pmod{4}$  or  $d \equiv 3 \pmod{4}$  and  $\left(\frac{c}{d}\right)$  is the extended Jacobi symbol. Also note that if  $h_1 \equiv h_2 \pmod{N}$

$$\Theta_\nu(A, h_1, N; \tau) = \Theta_\nu(A, h_2, N; \tau), \quad \Theta_\nu(A, -h, N; \tau) = (-1)^\nu \Theta_\nu(A, h, N; \tau).$$

**2.4. Indefinite theta functions.** We begin by defining (possibly indefinite) theta functions.

**Definition 1.** Let  $A \in M_m(\mathbb{Z})$  be a non-singular symmetric  $m \times m$  matrix,  $P : \mathbb{R}^m \rightarrow \mathbb{C}$  and  $a \in \mathbb{Q}^m$ . We define the associated theta function by  $(\tau = u + iv)$

$$\Theta_{A,P,a}(\tau) := \sum_{n \in a + \mathbb{Z}^m} P(\sqrt{v}n) q^{\frac{1}{2}n^T A n}.$$

The following theorem shows that under certain conditions  $\Theta_{A,P,a}$  is modular.

**Theorem 2.2** (Vignéras, [21]). *Suppose that  $A \in M_m(\mathbb{Z})$  is non-singular and that  $P$  satisfies the following conditions:*

- (1) *For any differential operator  $D$  of order two and any polynomial  $R$  of degree at most two, we have that  $D(w)(P(w)e^{\pi Q(w)})$  and  $R(w)P(w)e^{\pi Q(w)}$  belong to  $L^2(\mathbb{R}^m) \cap L^1(\mathbb{R}^m)$ .*
- (2) *Defining the Euler and Laplace operators ( $w := (w_1, \dots, w_m)^T$ ,  $\partial_w := (\frac{\partial}{\partial w_1}, \dots, \frac{\partial}{\partial w_m})^T$ )*

$$\mathcal{D} := w^T \partial_w \quad \text{and} \quad \Delta = \Delta_{A^{-1}} := \partial_w^T A^{-1} \partial_w,$$

*for some  $\lambda \in \mathbb{Z}$  the Vignéras differential equation holds:*

$$\left( \mathcal{D} - \frac{1}{4\pi} \Delta \right) P = \lambda P.$$

*Then, assuming that  $\Theta_{A,P,a}$  is absolutely locally convergent,  $\Theta_{A,P,a}$  is modular of weight  $\lambda + \frac{m}{2}$  for some subgroup of  $\mathrm{SL}_2(\mathbb{Z})$ .*

**2.5. Quantum modular forms.** We already motivated quantum modular forms in the introduction. The formal definition is as follows [26].

**Definition 2.** A function  $f : \mathcal{Q} \rightarrow \mathbb{C}$  (here  $\mathcal{Q} \subseteq \mathbb{Q}$ ) is called a *quantum modular form of weight  $k$  and multiplier  $\chi$  for a subgroup  $\Gamma$  of  $\mathrm{SL}_2(\mathbb{Z})$  and quantum set  $\mathcal{Q}$*  if for  $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ , the function

$$f(\tau) - \chi(M)^{-1}(c\tau + d)^{-k}f(M\tau)$$

can be extended to an open subset of  $\mathbb{R}$  and is real analytic there. We denote the vector space of such forms by  $\mathcal{Q}_k(\Gamma, \chi)$ .

*Remark.* Zagier also considered *strong quantum modular forms*. Here one is looking at asymptotic expansions instead of just values.

The introduction already gives examples of quantum modular forms. As mentioned there, the functions  $F_{j,p}$  satisfy modular type transformations making them quantum modular forms. More generally, for  $f \in S_k(\Gamma, \chi)$ , the space of cusp forms of weight  $k$  transforming as

$$f(M\tau) = (c\tau + d)^k \chi(M) f(\tau)$$

for  $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \subset \mathrm{SL}_2(\mathbb{Z})$  and  $\chi$  some multiplier, we set, for  $\frac{d}{c} \in \mathbb{Q}$ ,

$$I_f(\tau) := \int_{-\tau}^{i\infty} \frac{f(w)}{(-i(w + \tau))^{2-k}} dw, \quad r_{f, \frac{d}{c}}(\tau) := \int_{\frac{d}{c}}^{i\infty} \frac{f(w)}{(-i(w + \tau))^{2-k}} dw. \quad (2.11)$$

For weight  $k = \frac{1}{2}$ , we allow  $f \in M_{\frac{1}{2}}(\Gamma, \chi)$ , the space of holomorphic modular forms of weight  $\frac{1}{2}$ . To state the modularity properties of  $I_f$ , we let  $\Gamma^* := P\Gamma P^{-1}$ , where  $P := \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ .

**Lemma 2.3.** *We have the transformation, for  $M \in \Gamma^*$ ,*

$$I_f(\tau) - \chi^{-1}(M^*)(c\tau + d)^{k-2} I_f(M\tau) = r_{f, \frac{d}{c}}(\tau).$$

*The function  $I_f$  is defined on  $\mathbb{H} \cup \mathbb{Q}$  whereas  $r_{f, \frac{d}{c}}$  exists on all of  $\mathbb{R} \setminus \{-\frac{d}{c}\}$  and is real-analytic there. If  $f \in S_k(\Gamma, \chi)$ , then  $r_{f, \frac{d}{c}}$  exists on  $\mathbb{R}$ .*

**2.6. Higher Depth Quantum modular forms.** We next turn to generalizations of quantum modular forms.

**Definition 3.** A function  $f : \mathcal{Q} \rightarrow \mathbb{C}$  ( $\mathcal{Q} \subset \mathbb{Q}$ ) is called a *quantum modular form of depth  $N \in \mathbb{N}$ , weight  $k$ , multiplier  $\chi$ , and quantum set  $\mathcal{Q}$  for  $\Gamma$*  if for  $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$

$$f(\tau) - \chi(M)^{-1}(c\tau + d)^{-k} f(M\tau) \in \bigoplus_j \mathcal{Q}_{\kappa_j}^{N_j}(\Gamma, \chi_j) \mathcal{O}(R)$$

where  $j$  runs through a finite set,  $\kappa_j \in \frac{1}{2}\mathbb{Z}$ ,  $N_j \in \mathbb{N}$  with  $\max(N_j) = N-1$ ,  $\chi_j$  characters,  $\mathcal{O}(R)$  is the space of real-analytic functions on  $R \subset \mathbb{R}$  which contains an open subset of  $\mathbb{R}$ ,  $\mathcal{Q}_k^1(\Gamma, \chi) := \mathcal{Q}_k(\Gamma, \chi)$ ,  $\mathcal{Q}_k^0(\Gamma, \chi) := 1$ , and  $\mathcal{Q}_k^N(\Gamma, \chi)$  denotes the space of forms of weight  $k$ , depth  $N$ , multiplier  $\chi$  for  $\Gamma$ .

*Remark.* Again one can consider *higher depth strong quantum modular forms* by looking at asymptotic expansions instead of values. The examples of this paper satisfy this stronger property.

*Example.* For  $f_1 \in \mathcal{Q}_{k_1}^1(\Gamma_1, \chi_1)$  and  $f_2 \in \mathcal{Q}_{k_2}^1(\Gamma_2, \chi_2)$ ,  $f_1 f_2 \in \mathcal{Q}_{k_1+k_2}^2(\Gamma_1 \cap \Gamma_2, \chi_1 \chi_2)$ .

### 3. A RANK TWO FALSE THETA FUNCTION

We briefly recall a construction from [5, 7]. For  $p \in \mathbb{N}_{\geq 2}$ , there is a vertex operator algebra  $W(p)_{A_2}$  associated to the simple Lie algebra  $\mathfrak{sl}_3$  (more precisely, its root lattice of type  $A_2$ ), whose proposed full character  $\text{ch}[W(p)_{A_2}](\tau, z)$  [9], [5, 7] (see also [2]) satisfies

$$\begin{aligned} \eta(\tau)^2 \text{ch}[W(p)_{A_2}](\tau, z) &= \sum_{m_1, m_2 \in \mathbb{Z}} \frac{q^{p \left( \left(m_1 - \frac{1}{p}\right)^2 + \left(m_2 - \frac{1}{p}\right)^2 - \left(m_1 - \frac{1}{p}\right) \left(m_2 - \frac{1}{p}\right) \right)}}{(1 - z_1^{-1})(1 - z_2^{-1})(1 - z_1^{-1} z_2^{-1})} \left( z_1^{m_1-1} z_2^{m_2-1} - z_1^{-m_1+m_2-1} \right. \\ &\quad \times \left. z_2^{m_2-1} - z_1^{m_1-1} z_2^{-m_2+m_1-1} + z_1^{-m_2-1} z_2^{-m_2+m_1-1} + z_1^{-m_1+m_2-1} z_2^{-m_1-1} - z_1^{-m_2-1} z_2^{-m_1-1} \right). \end{aligned}$$

The six term expression in the numerator comes from summation over the Weyl group  $W \cong S_3$  of  $\mathfrak{sl}_3$ . Thanks to Weyl's character formula, the rational  $z$ -part is in fact a Laurent polynomial. There are two important operations on this character:

- (1) taking the limit  $z = (z_1, z_2) \rightarrow (1, 1)$ , yielding a modular form [5];
- (2) taking the constant term

$$\text{ch}[W^0(p)_{A_2}](\tau) := \text{CT}_{z_1, z_2} \text{ch}[W(p)_{A_2}](\tau, z),$$

which computes the character of another vertex algebra. It was shown in [5] that

$$\text{ch}[W^0(p)_{A_2}](\tau) = \frac{1}{\eta(\tau)^2} F(q).$$



Note that formulas like  $\eta(\tau)^{\text{rank}(Q)} \text{ch}[W^0(p)_Q](\tau)$ , where  $Q$  is any root lattice [5, 7], are of interest beyond vertex algebra theory. The coefficients appearing in the  $q$ -expansion are essentially dimensions of the zero weight spaces of finite-dimensional irreducible representations of simple Lie algebras (for the recent progress in understanding these numbers see [17]).

*Remark.* Modular-type properties of regularized (or Jacobi) characters, in particular  $\text{ch}[W^0(p)_{A_2}^\varepsilon](\tau)$ , were investigated in [7] (see also [6]). There are two important differences between the current work and [7]. In this paper, the value of the Jacobi parameter  $\varepsilon$  is always zero whereas in [7] it is necessarily non-zero. Secondly, there seems to be no clear connection between transformation formulas appearing in [7] and mock modular forms. On the other hand, here we make this connection quite explicit by virtue of generalized Eichler integrals (see Section 5).

Let  $n_1 = m_1 - m_2, n_2 = m_2$  in (1.7) and then change  $n_1 \mapsto 3n_1$ . Then we have,  $F$  as in (1.7),

$$F(q) = f_1(q) + f_2(q) + f_3(q),$$

where, with  $Q(x_1, x_2) := 3x_1^2 + 3x_1x_2 + x_2^2$ , we define

$$\begin{aligned} f_1(q) &:= q^{\frac{1}{p}} \sum_{n_1, n_2 \geq 0}^* n_2 q^{pQ(n_1, n_2)} (q^{-3n_1-2n_2} - q^{3n_1+2n_2}), \\ f_2(q) &:= q^{\frac{1}{p}} \sum_{n_1, n_2 \geq 0}^* n_2 q^{pQ(n_1, n_2)} (q^{n_2} - q^{-n_2}), \\ f_3(q) &:= q^{\frac{1}{p}} \sum_{n_1, n_2 \geq 0}^* n_2 q^{pQ(n_1, n_2)} (q^{3n_1+n_2} - q^{-3n_1-n_2}). \end{aligned}$$

Here  $\sum^*$  means that the  $n_1 = 0$  term is weighted by  $\frac{1}{2}$ . We then rewrite

$$\begin{aligned} f_1(q) &= - \sum_{n_1, n_2 \geq 0} \left( n_2 + \frac{1}{p} \right) q^{pQ(n_1+1, n_2+\frac{1}{p})} + \sum_{n_1, n_2 \geq 0} \left( n_2 + 1 - \frac{1}{p} \right) q^{pQ(n_1, n_2+1-\frac{1}{p})} \\ &\quad + \frac{1}{p} \sum_{n_1, n_2 \geq 0} q^{pQ(n_1+1, n_2+\frac{1}{p})} + \frac{1}{p} \sum_{n_1, n_2 \geq 0} q^{pQ(n_1, n_2+1-\frac{1}{p})} - \frac{1}{2} \sum_{m \geq 0} \left( m + \frac{1}{p} \right) q^{p(m+\frac{1}{p})^2} \\ &\quad - \frac{1}{2} \sum_{m \geq 0} \left( m + 1 - \frac{1}{p} \right) q^{p(m+1-\frac{1}{p})^2} + \frac{1}{2p} \sum_{m \geq 0} q^{p(m+\frac{1}{p})^2} - \frac{1}{2p} \sum_{m \geq 0} q^{p(m+1-\frac{1}{p})^2}, \\ f_2(q) &= \sum_{n_1, n_2 \geq 0} \left( n_2 + \frac{2}{p} \right) q^{pQ(n_1+1-\frac{1}{p}, n_2+\frac{2}{p})} - \sum_{n_1, n_2 \geq 0} \left( n_2 + 1 - \frac{2}{p} \right) q^{pQ(n_1+\frac{1}{p}, n_2+1-\frac{2}{p})} \\ &\quad - \frac{2}{p} \sum_{n_1, n_2 \geq 0} q^{pQ(n_1+1-\frac{1}{p}, n_2+\frac{2}{p})} - \frac{2}{p} \sum_{n_1, n_2 \geq 0} q^{pQ(n_1+\frac{1}{p}, n_2+1-\frac{2}{p})} \\ &\quad + \frac{q^{\frac{3}{4p}}}{2} \sum_{m \geq 1} m q^{p(m+1-\frac{1}{2p})^2} + \frac{q^{\frac{3}{4p}}}{2} \sum_{m \geq 1} m q^{p(m+\frac{1}{2p})^2}, \\ f_3(q) &= \sum_{n_1, n_2 \geq 0} \left( n_2 + 1 - \frac{1}{p} \right) q^{pQ(n_1+\frac{1}{p}, n_2+1-\frac{1}{p})} - \sum_{n_1, n_2 \geq 0} \left( n_2 + \frac{1}{p} \right) q^{pQ(n_1+1-\frac{1}{p}, n_2+\frac{1}{p})} \end{aligned}$$

$$\begin{aligned}
& + \frac{1}{p} \sum_{n_1, n_2 \geq 0} q^{pQ\left(n_1 + \frac{1}{p}, n_2 + 1 - \frac{1}{p}\right)} + \frac{1}{p} \sum_{n_1, n_2 \geq 0} q^{pQ\left(n_1 + 1 - \frac{1}{p}, n_2 + \frac{1}{p}\right)} \\
& - \frac{q^{\frac{3}{4p}}}{2} \sum_{m \geq 1} m q^{p\left(m + \frac{1}{2p}\right)^2} - \frac{q^{\frac{3}{4p}}}{2} \sum_{m \geq 1} m q^{p\left(m + 1 - \frac{1}{2p}\right)^2}.
\end{aligned}$$

We thus obtain

$$f(q) = \frac{1}{p} F_1(q^p) + F_2(q^p)$$

with

$$F_1(q) := \sum_{\alpha \in \mathcal{S}} \varepsilon(\alpha) \sum_{n \in \alpha + \mathbb{N}_0^2} q^{Q(n)} + \frac{1}{2} \sum_{m \in \mathbb{Z}} \operatorname{sgn}\left(m + \frac{1}{p}\right) q^{\left(m + \frac{1}{p}\right)^2}, \quad (3.1)$$

where

$$\mathcal{S} := \left\{ \left(1 - \frac{1}{p}, \frac{2}{p}\right), \left(\frac{1}{p}, 1 - \frac{2}{p}\right), \left(1, \frac{1}{p}\right) \left(0, 1 - \frac{1}{p}\right), \left(\frac{1}{p}, 1 - \frac{1}{p}\right), \left(1 - \frac{1}{p}, \frac{1}{p}\right) \right\},$$

and for  $\alpha \pmod{1}$ , we set

$$\varepsilon(\alpha) := \begin{cases} -2 & \text{if } \alpha \in \left\{ \left(1 - \frac{1}{p}, \frac{2}{p}\right), \left(\frac{1}{p}, 1 - \frac{2}{p}\right) \right\}, \\ 1 & \text{otherwise.} \end{cases}$$

Moreover

$$F_2(q) := \sum_{\alpha \in \mathcal{S}} \eta(\alpha) \sum_{n \in \alpha + \mathbb{N}_0^2} n_2 q^{Q(n)} - \frac{1}{2} \sum_{m \in \mathbb{Z}} \left| m + \frac{1}{p} \right| q^{\left(m + \frac{1}{p}\right)^2}, \quad (3.2)$$

where for  $\alpha \pmod{1}$ , we let

$$\eta(\alpha) := \begin{cases} 1 & \text{if } \alpha \in \left\{ \left(1 - \frac{1}{p}, \frac{2}{p}\right), \left(0, 1 - \frac{1}{p}\right), \left(\frac{1}{p}, 1 - \frac{1}{p}\right) \right\}, \\ -1 & \text{otherwise.} \end{cases}$$

#### 4. ASYMPTOTIC BEHAVIOR OF $F_1$ AND $F_2$

In this section we determine the asymptotic behavior of  $F(e^{2\pi i \frac{h}{k} - t})$  ( $h, k \in \mathbb{Z}$  with  $k > 0$  and  $\gcd(h, k) = 1$ ) as  $t \rightarrow 0^+$ .

**4.1. The function  $F_1$ .** We decompose

$$F_1(q) = F_{1,1}(q) + F_{1,2}(q),$$

where

$$F_{1,1}(q) := \sum_{\alpha \in \mathcal{S}} \varepsilon(\alpha) \sum_{n \in \alpha + \mathbb{N}_0^2} q^{Q(n)}, \quad F_{1,2}(q) := \frac{1}{2} \sum_{m \in \frac{1}{p} + \mathbb{Z}} \operatorname{sgn}(m) q^{m^2}.$$

We first study the asymptotic behavior of  $F_{1,1}$ , rewriting it in a shape in which we can apply the Euler-Maclaurin formula (2.8). For this, let  $n \mapsto \ell + n \frac{kp}{\delta}$  with  $n \in \mathbb{N}_0^2$ ,  $0 \leq \ell \leq \frac{kp}{\delta} - 1$ , where

$\delta := \gcd(h, p)$ . Here by the inequality we mean that it should hold componentwise. It is not hard to see that, with  $\mathcal{F}_1(x) := e^{-Q(x)}$ ,

$$F_{1,1} \left( e^{2\pi i \frac{h}{k} t} \right) = \sum_{\alpha \in \mathcal{S}} \varepsilon(\alpha) \sum_{0 \leq \ell \leq \frac{kp}{\delta} - 1} e^{2\pi i \frac{h}{k} Q(\ell + \alpha)} \sum_{n \in \frac{\delta}{kp}(\ell + \alpha) + \mathbb{N}_0^2} \mathcal{F}_1 \left( \frac{kp}{\delta} \sqrt{t} n \right).$$

The main term in (2.8) is then

$$\frac{\delta^2}{k^2 p^2 t} \mathcal{I}_{\mathcal{F}_1} \sum_{\alpha \in \mathcal{S}} \varepsilon(\alpha) \sum_{0 \leq \ell \leq \frac{kp}{\delta} - 1} e^{2\pi i \frac{h}{k} Q(\ell + \alpha)}. \quad (4.1)$$

It is not hard to see that one may let  $\ell$  run  $(\bmod \frac{kp}{\delta})$  (again meant componentwise). Write  $\ell = n + k\nu$  with  $n \pmod k$ ,  $\nu \pmod{\frac{p}{\delta}}$ , and  $a \in \{(-1, 2), (1, -2), (0, 1), (0, -1), (1, -1), (-1, 1)\}$  such that  $\alpha - \frac{a}{p} \in \mathbb{Z}^2$ . We then compute that the sum on  $\ell$  in (4.1) equals (since  $Q(a) = 1$ )

$$\begin{aligned} e^{2\pi i \frac{h}{p^2 k}} \sum_{n \pmod k} e^{\frac{2\pi i h}{pk} (3(pn_1^2 + 2a_1 n_1) + 3(pn_1 n_2 + a_2 n_1 + a_1 n_2) + pn_2^2 + 2a_2 n_2)} \\ \times \sum_{\nu \pmod{\frac{p}{\delta}}} e^{\frac{2\pi i h/\delta}{p/\delta} ((6a_1 + 3a_2)\nu_1 + (2a_2 + 3a_1)\nu_2)}. \end{aligned}$$

Since  $\gcd(\frac{h}{\delta}, \frac{p}{\delta}) = 1$ , the inner sum vanishes unless  $\frac{p}{\delta} \mid 3(2a_1 + a_2)$  and  $\frac{p}{\delta} \mid (2a_2 + 3a_1)$ . If  $3 \mid \frac{p}{\delta}$ , then in particular  $3 \mid a_2$ . This is however not satisfied for elements in  $\mathcal{S}$ . If  $3 \nmid \frac{p}{\delta}$ , then we easily obtain that  $a_1 \equiv a_2 \equiv 0 \pmod{\frac{p}{\delta}}$ , implying that  $\frac{p}{\delta} = 1$ . We are thus left to show that  $(\frac{p}{\delta} = 1)$

$$\sum_{\alpha \in \mathcal{S}} \varepsilon(\alpha) \sum_{n \pmod k} e^{\frac{2\pi i h/\delta}{k} (3(pn_1^2 + 2a_1 n_1) + 3(pn_1 n_2 + a_2 n_1 + a_1 n_2) + pn_2^2 + 2a_2 n_2)} = 0. \quad (4.2)$$

Changing  $n \mapsto n - a\bar{p}$  with  $\bar{p}$  the inverse of  $p \pmod k$  (note that  $\frac{p}{\delta} = 1$  implies that  $\gcd(p, k) = 1$ ) the sum on  $n$  equals

$$e^{-\frac{2\pi i \bar{p} h/\delta}{k}} \sum_{n \pmod k} e^{\frac{2\pi i h}{k} Q(n)},$$

independent of  $a$ . Thus (4.2) holds.

The second term in (2.8) is

$$- \sum_{\alpha \in \mathcal{S}} \varepsilon(\alpha) \sum_{0 \leq \ell \leq \frac{kp}{\delta} - 1} e^{2\pi i \frac{h}{k} Q(\ell + \alpha)} \sum_{n_2 \geq 0} \frac{B_{n_2+1} \left( \frac{\delta(\ell_2 + \alpha_2)}{kp} \right)}{(n_2 + 1)!} \int_0^\infty \mathcal{F}_1^{(0, n_2)}(x_1, 0) dx_1 \left( \frac{kp\sqrt{t}}{\delta} \right)^{n_2-1}.$$

We claim that the terms with  $n_2$  even vanish. This follows, once we show that for  $\alpha \in \mathcal{S}$

$$\sum_{0 \leq \ell \leq \frac{kp}{\delta} - 1} \left( e^{2\pi i \frac{h}{k} Q(\ell + \alpha)} B_{2n_2+1} \left( \frac{\delta(\ell_2 + \alpha_2)}{kp} \right) + e^{2\pi i \frac{h}{k} Q(\ell + 1 - \alpha)} B_{2n_2+1} \left( \frac{\delta(\ell_2 + 1 - \alpha_2)}{kp} \right) \right) = 0.$$

This is seen to be true by the change of variables  $\ell \mapsto -\ell + (-1 + \frac{k}{p})(1, 1)$  for the second term.

Arguing in the same way for  $n_2$  odd, we obtain

$$-2 \sum_{\alpha \in \mathcal{S}^*} \varepsilon(\alpha) \sum_{0 \leq \ell \leq \frac{kp}{\delta} - 1} e^{2\pi i \frac{h}{k} Q(\ell + \alpha)} \sum_{n_2 \geq 0} \frac{B_{2n_2+2} \left( \frac{\delta(\ell_2 + \alpha_2)}{kp} \right)}{(2n_2 + 2)!} \int_0^\infty \mathcal{F}_1^{(0, 2n_2+1)}(x_1, 0) dx_1 \left( \frac{k^2 p^2}{\delta^2} t \right)^{n_2},$$

where

$$\mathcal{S}^* := \left\{ \left( 1 - \frac{1}{p}, \frac{2}{p} \right), \left( 0, 1 - \frac{1}{p} \right), \left( \frac{1}{p}, 1 - \frac{1}{p} \right) \right\}.$$

The third term in (2.8) is treated in the same way, yielding

$$-2 \sum_{\alpha \in \mathcal{S}^*} \varepsilon(\alpha) \sum_{0 \leq \ell \leq \frac{kp}{\delta} - 1} e^{2\pi i \frac{h}{k} Q(\ell + \alpha)} \sum_{n_1 \geq 0} \frac{B_{2n_1+2} \left( \frac{\delta(\ell_1 + \alpha_1)}{kp} \right)}{(2n_1 + 2)!} \int_0^\infty \mathcal{F}_1^{(2n_1+1, 0)}(0, x_2) dx_2 \left( \frac{k^2 p^2}{\delta^2} t \right)^{n_1}.$$

The final term in (2.8) equals

$$\begin{aligned} & \sum_{\alpha \in \mathcal{S}} \varepsilon(\alpha) \sum_{0 \leq \ell \leq \frac{kp}{\delta} - 1} e^{2\pi i \frac{h}{k} Q(\ell + \alpha)} \\ & \times \sum_{n_1, n_2 \geq 0} \frac{B_{n_1+1} \left( \frac{\delta(\ell_1 + \alpha_1)}{kp} \right)}{(n_1 + 1)!} \frac{B_{n_2+1} \left( \frac{\delta(\ell_2 + \alpha_2)}{kp} \right)}{(n_2 + 1)!} \mathcal{F}_1^{(n_1, n_2)}(0, 0) \left( \frac{kp\sqrt{t}}{\delta} \right)^{n_1 + n_2}. \end{aligned}$$

Arguing in the same way as before this equals

$$\begin{aligned} & 2 \sum_{\alpha \in \mathcal{S}^*} \varepsilon(\alpha) \sum_{0 \leq \ell \leq \frac{kp}{\delta} - 1} e^{2\pi i \frac{h}{k} Q(\ell + \alpha)} \\ & \times \sum_{\substack{n_1, n_2 \geq 0 \\ n_1 \equiv n_2 \pmod{2}}} \frac{B_{n_1+1} \left( \frac{\delta(\ell_1 + \alpha_1)}{kp} \right)}{(n_1 + 1)!} \frac{B_{n_2+1} \left( \frac{\delta(\ell_2 + \alpha_2)}{kp} \right)}{(n_2 + 1)!} \mathcal{F}_1^{(n_1, n_2)}(0, 0) \left( \frac{kp\sqrt{t}}{\delta} \right)^{n_1 + n_2}. \end{aligned}$$

The function  $F_{1,2}$  is treated similarly, yielding, with  $\mathcal{F}_2(x) := e^{-x^2}$ ,

$$- \sum_{0 \leq r \leq \frac{kp}{\delta} - 1} e^{2\pi i \frac{h}{k} \left( r + \frac{1}{p} \right)^2} \sum_{m \geq 0} \frac{B_{2m+1} \left( \frac{\delta \left( r + \frac{1}{p} \right)}{kp} \right)}{(2m + 1)!} \mathcal{F}_2^{(2m)}(0) \left( \frac{k^2 p^2}{\delta^2} t \right)^m.$$

**4.2. The function  $F_2$ .** Since the calculations are similar to those for  $F_1$ , we skip some of the details. We throughout assume that  $\gcd(p, h) = 1$ . Decompose

$$F_2(q) = F_{2,1}(q) + F_{2,2}(q),$$

with

$$F_{2,1}(q) := \sum_{\alpha \in \mathcal{S}} \eta(\alpha) \sum_{n \in \alpha + \mathbb{N}_0^2} n_2 q^{Q(n)}, \quad F_{2,2}(q) := -\frac{1}{2} \sum_{m \in \frac{1}{p} + \mathbb{Z}} |m| q^{m^2}.$$

We first study the asymptotic behavior of  $F_{2,1}$ . Arguing as for  $F_{1,1}$ , this is

$$\frac{1}{\sqrt{t}} \sum_{\alpha \in \mathcal{S}} \eta(\alpha) \sum_{0 \leq \ell \leq kp-1} e^{2\pi i \frac{h}{k} Q(\ell+\alpha)} \sum_{n \in \frac{1}{kp}(\ell+\alpha) + \mathbb{N}_0^2} \mathcal{G}_1(kp\sqrt{t}n),$$

with  $\mathcal{G}_1(x) := x_2 \mathcal{F}_1(x)$ . The Euler-Maclaurin main term is

$$\frac{kp}{(kp\sqrt{t})^3} \mathcal{I}_{\mathcal{G}_1} \sum_{\alpha \in \mathcal{S}} \eta(\alpha) \sum_{\ell \pmod{kp}} e^{2\pi i \frac{h}{k} Q(\ell+\alpha)}.$$

As in Subsection 4.1, one can show that this vanishes.

The second term in the Euler-Maclaurin summation formula is

$$-2 \sum_{\alpha \in \mathcal{S}^*} \sum_{0 \leq \ell \leq kp-1} e^{2\pi i \frac{h}{k} Q(\ell+\alpha)} \sum_{n_2 \geq 1} \frac{B_{2n_2+1} \left( \frac{\ell_2+\alpha_2}{kp} \right)}{(2n_2+1)!} \int_0^\infty \mathcal{G}_1^{(0,2n_2)}(x_1, 0) dx_1 (kp)^{2n_2-1} t^{n_2-1},$$

again pairing  $\alpha$  and  $1-\alpha$  and using that  $\mathcal{G}_1(x_1, 0) = 0$ .

The third term in the Euler-Maclaurin summation formula is

$$- \sum_{\alpha \in \mathcal{S}} \eta(\alpha) \sum_{0 \leq \ell \leq kp-1} e^{2\pi i \frac{h}{k} Q(\ell+\alpha)} \sum_{n_1 \geq 0} \frac{B_{n_1+1} \left( \frac{\ell_1+\alpha_1}{kp} \right)}{(n_1+1)!} \int_0^\infty \mathcal{G}_1^{(n_1,0)}(0, x_2) dx_2 (kp)^{n_1-1} t^{\frac{n_1}{2}-1}.$$

We claim that under the assumption  $\gcd(h, p) = 1$  this term vanishes. Noting that we can let  $\ell_2$  run  $\pmod{kp}$ , we split  $\ell_2 = r_1 + kr_2$  with  $r_1 \pmod{k}$ ,  $r_2 \pmod{p}$ . With  $\alpha = \frac{a}{p}$ , the sum on  $r_2$  is

$$\sum_{r_2 \pmod{p}} e^{2\pi i \frac{hr_2}{p} (3a_1+2a_2)} = 0$$

since  $p \nmid (3a_1 + 2a_2)h$ .

The final term in Euler-Maclaurin is

$$2 \sum_{\alpha \in \mathcal{S}^*} \sum_{0 \leq \ell \leq kp-1} e^{2\pi i \frac{h}{k} Q(\ell+\alpha)} \sum_{\substack{n_1, n_2 \geq 0 \\ n_1 \not\equiv n_2 \pmod{2}}} \frac{B_{n_1+1} \left( \frac{\ell_1+\alpha_1}{kp} \right)}{(n_1+1)!} \frac{B_{n_2+1} \left( \frac{\ell_2+\alpha_2}{kp} \right)}{(n_2+1)!} \mathcal{G}_1^{(n_1, n_2)}(0, 0) (kp)^{n_1+n_2} t^{n_1+n_2-1},$$

again pairing  $\alpha$  with  $1-\alpha$ .

We next turn to  $F_{2,2}$ . The Euler-Maclaurin main term is, with  $\mathcal{G}_2(x) := x \mathcal{F}_2(x)$ ,

$$-\frac{1}{kpt} \mathcal{I}_{\mathcal{G}_2} \sum_{r \pmod{kp}} e^{2\pi i \frac{h}{k} \left(r + \frac{1}{p}\right)^2}.$$

Writing  $r = r_1 + kr_2$  with  $r_1 \pmod{k}$ ,  $r_2 \pmod{p}$ , the sum on  $r_2$  becomes, using that  $p \neq 2$ ,

$$\sum_{r_2 \pmod{p}} e^{\frac{4\pi i hr_2}{p}} = 0. \quad (4.3)$$

Arguing as before, the second term in the Euler-Maclaurin summation formula is

$$\sum_{0 \leq r \leq kp-1} e^{2\pi i \frac{h}{k} \left(r + \frac{1}{p}\right)^2} \sum_{m \geq 0} \frac{B_{2m+2} \left(\frac{\ell + \frac{1}{p}}{kp}\right)}{(2m+2)!} \mathcal{G}_2^{(2m+1)}(0) (kp)^{2m+1} t^m.$$

## 5. COMPANIONS IN THE LOWER HALF PLANE

In this section we investigate multivariable Eichler integrals.

**5.1. Multiple Eichler integrals.** Let  $f_j \in S_{k_j}(\Gamma, \chi_j)$ ; if  $k_j = \frac{1}{2}$  we also allow  $f_j \in M_{\frac{1}{2}}(\Gamma, \chi_j)$ . Define the *double Eichler integral*

$$I_{f_1, f_2}(\tau) := \int_{-\bar{\tau}}^{i\infty} \int_{w_1}^{i\infty} \frac{f_1(w_1) f_2(w_2)}{(-i(w_1 + \tau))^{2-k_1} (-i(w_2 + \tau))^{2-k_2}} dw_2 dw_1,$$

and the *multiple error of modularity*

$$r_{f_1, f_2, \frac{d}{c}}(\tau) := \int_{\frac{d}{c}}^{i\infty} \int_{w_1}^{\frac{d}{c}} \frac{f_1(w_1) f_2(w_2)}{(-i(w_1 + \tau))^{2-k_1} (-i(w_2 + \tau))^{2-k_2}} dw_2 dw_1.$$

**Theorem 5.1.** *We have, for  $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma^*$ ,*

$$I_{f_1, f_2}(\tau) - \chi_1^{-1}(M^*) \chi_2^{-1}(M^*) (c\tau + d)^{k_1+k_2-4} I_{f_1, f_2}(M\tau) = r_{f_1, f_2, \frac{d}{c}}(\tau) + I_{f_1}(\tau) r_{f_2, \frac{d}{c}}(\tau). \quad (5.1)$$

Moreover  $r_{f_1, f_2, \frac{d}{c}} \in \mathcal{O}(\mathbb{R} \setminus \{-\frac{d}{c}\})$ . If  $f_j \in S_{k_j}(\Gamma, \chi_j)$  (for  $j = 1, 2$ ), then  $r_{f_1, f_2, \frac{d}{c}} \in \mathcal{O}(\mathbb{R})$ .

*Proof of Theorem 5.1.* For simplicity, we assume that  $\frac{1}{2} \leq k_j \leq 2$  and that  $f_1, f_2$  are cuspidal. A direct calculation gives that, for  $M \in \Gamma^*$ ,

$$I_{f_1, f_2}(M\tau) = \chi_1(M^*) \chi_2(M^*) (c\tau + d)^{4-k_1-k_2} \int_{-\bar{\tau}}^{\frac{d}{c}} \int_{w_1}^{\frac{d}{c}} \frac{f_1(w_1) f_2(w_2)}{(-i(w_1 + \tau))^{2-k_1} (-i(w_2 + \tau))^{2-k_2}} dw_2 dw_1.$$

The transformation (5.1) now follows by splitting

$$\int_{-\bar{\tau}}^{\frac{d}{c}} \int_{w_1}^{\frac{d}{c}} = \int_{-\bar{\tau}}^{i\infty} \int_{w_1}^{i\infty} + \int_{\frac{d}{c}}^{i\infty} \int_{\frac{d}{c}}^{w_1} - \int_{-\bar{\tau}}^{i\infty} \int_{\frac{d}{c}}^{i\infty}.$$

Using Lemma 2.3, we are left to show that  $r_{f_1, f_2, \frac{d}{c}}$  is real-analytic on  $\mathbb{R}$  which follows once we prove that the following function is real-analytic

$$\int_0^\infty \int_0^{w_1} \frac{f_1\left(iw_1 + \frac{d}{c}\right) f_2\left(iw_2 + \frac{d}{c}\right)}{\left(w_1 - i\left(\tau + \frac{d}{c}\right)\right)^{2-k_1} \left(w_2 - i\left(\tau + \frac{d}{c}\right)\right)^{2-k_2}} dw_2 dw_1. \quad (5.2)$$

We use that for  $w_j \geq 1$

$$f_j\left(iw_j + \frac{d}{c}\right) \ll e^{-a_j w_j} \quad a_j \in \mathbb{R}^+, \quad (5.3)$$

and for  $0 < w_j \leq 1$  (the implied constant and  $b_j$  may depend on  $c$ )

$$f_j\left(iw_j + \frac{d}{c}\right) \ll w_j^{-k_j} e^{-\frac{b_j}{w_j}} \quad b_j \in \mathbb{R}^+. \quad (5.4)$$

To show real-analyticity of (5.2) on  $\mathbb{R}$ , we split it into 3 pieces. Firstly, set

$$I_1 := \int_1^\infty \int_1^{w_1} \frac{f_1\left(iw_1 + \frac{d}{c}\right) f_2\left(iw_2 + \frac{d}{c}\right)}{\left(w_1 - i\left(\tau + \frac{d}{c}\right)\right)^{2-k_1} \left(w_2 - i\left(\tau + \frac{d}{c}\right)\right)^{2-k_2}} dw_2 dw_1.$$

Using (5.3) and that  $w_1 \geq 1$  easily gives the locally uniform bound

$$I_1 \ll \int_1^\infty \frac{e^{-a_1 w_1}}{w_1^{2-k_1}} dw_1 \int_1^\infty \frac{e^{-a_2 w_2}}{w_2^{2-k_2}} dw_2 \ll 1.$$

Next consider

$$I_2 := \int_0^1 \int_0^{w_1} \frac{f_1\left(iw_1 + \frac{d}{c}\right) f_2\left(iw_2 + \frac{d}{c}\right)}{\left(w_1 - i\left(\tau + \frac{d}{c}\right)\right)^{2-k_1} \left(w_2 - i\left(\tau + \frac{d}{c}\right)\right)^{2-k_2}} dw_2 dw_1.$$

Using (5.4) gives that

$$I_2 \ll \int_0^1 \frac{e^{-\frac{b_1}{w_1}}}{w_1^{2-k_1}} dw_1 \int_0^1 \frac{e^{-\frac{b_2}{w_2}}}{w_2^{2-k_2}} dw_2 \ll 1.$$

Finally, we set

$$I_3 := \int_1^\infty \frac{f_1\left(iw_1 + \frac{d}{c}\right)}{\left(w_1 - i\left(\tau + \frac{d}{c}\right)\right)^{2-k_1}} dw_1 \int_0^1 \frac{f_2\left(iw_2 + \frac{d}{c}\right)}{\left(w_2 - i\left(\tau + \frac{d}{c}\right)\right)^{2-k_2}} dw_2.$$

Combining the above bounds gives again  $I_3 \ll 1$ . □

**5.2. Special multiple Eichler integrals of weight one.** Define for  $\alpha \in \mathcal{S}^*$

$$\mathcal{E}_{1,\alpha}(\tau) := -\frac{\sqrt{3}}{4} \int_{-\bar{\tau}}^{i\infty} \int_{w_1}^{i\infty} \frac{\theta_1(\alpha; w_1, w_2) + \theta_2(\alpha; w_1, w_2)}{\sqrt{-i(w_1 + \tau)} \sqrt{-i(w_2 + \tau)}} dw_2 dw_1$$

with

$$\theta_1(\alpha; w_1, w_2) := \sum_{n \in \alpha + \mathbb{Z}^2} (2n_1 + n_2) n_2 e^{\frac{3\pi i}{2}(2n_1 + n_2)^2 w_1 + \frac{\pi i n_2^2 w_2}{2}},$$

$$\theta_2(\alpha; w_1, w_2) := \sum_{n \in \alpha + \mathbb{Z}^2} (3n_1 + 2n_2) n_1 e^{\frac{\pi i}{2}(3n_1 + 2n_2)^2 w_1 + \frac{3\pi i n_1^2 w_2}{2}}.$$

Moreover set

$$\mathcal{E}_1(\tau) := \sum_{\alpha \in \mathcal{S}^*} \varepsilon(\alpha) \mathcal{E}_{1,\alpha}(p\tau),$$

$$\Gamma_p := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(24p) : b \equiv 0 \pmod{4p}, d \equiv \pm 1 \pmod{2p} \right\}.$$

*Remark.* Note that  $\Gamma_p^* = \Gamma_p$ .

**Proposition 5.2.** *We have, for  $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_p$ ,*

$$\mathcal{E}_1(\tau) - \left(\frac{-3}{d}\right) (c\tau + d)^{-1} \mathcal{E}_1(M\tau) = \sum_{j=1}^{12} \left( r_{f_j, g_j, \frac{d}{c}}(\tau) + I_{f_j}(\tau) r_{g_j, \frac{d}{c}}(\tau) \right),$$

where  $f_j, g_j$  are cusp forms of weight  $\frac{3}{2}$  (with some multiplier).

*Proof.* To use Theorem 5.1, we write  $\theta_j$  in terms of Shimura's theta functions (2.9). For  $\theta_1$ , we set  $\nu_1 := 2n_1 + n_2$ ,  $\nu_2 := n_2$ . Then  $\nu_1 \in 2\alpha_1 + \alpha_2 + \mathbb{Z}$ ,  $\nu_2 \in \alpha_2 + \mathbb{Z}$ ,  $\nu_1 - \nu_2 \in 2\alpha_1 + 2\mathbb{Z}$ . Thus

$$\begin{aligned} \theta_1(\alpha; w_1, w_2) &= \sum_{\substack{\nu \in (2\alpha_1 + \alpha_2, \alpha_2) + \mathbb{Z}^2 \\ \nu_1 - \nu_2 \in 2\alpha_1 + 2\mathbb{Z}}} \nu_1 \nu_2 e^{\frac{3\pi i \nu_1^2 w_1}{2} + \frac{\pi i \nu_2^2 w_2}{2}} \\ &= \sum_{\varrho \in \{0, 1\}} \sum_{\nu_1 \in 2\alpha_1 + \alpha_2 + \varrho + 2\mathbb{Z}} \nu_1 e^{\frac{3\pi i \nu_1^2 w_1}{2}} \sum_{\nu_2 \in \alpha_2 + \varrho + 2\mathbb{Z}} \nu_2 e^{\frac{\pi i \nu_2^2 w_2}{2}}. \end{aligned}$$

Summing up then easily gives

$$\begin{aligned} \sum_{\alpha \in \mathcal{S}^*} \varepsilon(\alpha) \theta_1(\alpha; w_1, w_2) &= \frac{1}{p^2} \sum_{A \in \mathcal{A}} \varepsilon_1(A) \sum_{\nu_1 \equiv A_1 \pmod{2p}} \nu_1 e^{\frac{3\pi i \nu_1^2 w_1}{2p^2}} \sum_{\nu_2 \equiv A_2 \pmod{2p}} \nu_2 e^{\frac{\pi i \nu_2^2 w_2}{2p^2}} \\ &= \frac{1}{p^2} \sum_{A \in \mathcal{A}} \varepsilon_1(A) \Theta_1\left(2p, A_1, 2p; \frac{3w_1}{p}\right) \Theta_1\left(2p, A_2, 2p; \frac{w_2}{p}\right) \end{aligned}$$

with

$$\mathcal{A} := \{(0, 2), (p, p+2), (p-1, p-1), (-1, -1), (p+1, p-1), (1, -1)\}, \varepsilon_1(A) := \varepsilon\left(\frac{A_1 - A_2}{2}, A_2\right).$$

For  $\theta_2$ , we proceed similarly. Set  $\nu_1 = 3n_1 + 2n_2$ ,  $\nu_2 = n_1$ . Then  $\nu_1 \in 3\alpha_1 + 2\alpha_2 + \mathbb{Z}$ ,  $\nu_2 \in \alpha_1 + \mathbb{Z}$ ,  $\nu_1 - 3\nu_2 \in 2\alpha_2 + 2\mathbb{Z}$ . Thus

$$\begin{aligned} \theta_2(\alpha; w_1, w_2) &= \sum_{\substack{\nu \in (3\alpha_1 + 2\alpha_2, \alpha_1) + \mathbb{Z}^2 \\ \nu_1 - 3\nu_2 \in 2\alpha_2 + 2\mathbb{Z}}} \nu_1 \nu_2 e^{\frac{\pi i \nu_1^2 w_1}{2} + \frac{3\pi i \nu_2^2 w_2}{2}} \\ &= \sum_{\varrho \in \{0, 1\}} \sum_{\nu_1 \in 3\alpha_1 + 2\alpha_2 + \varrho + 2\mathbb{Z}} \nu_1 e^{\frac{\pi i \nu_1^2 w_1}{2}} \sum_{\nu_2 \in \alpha_1 + \varrho + 2\mathbb{Z}} \nu_2 e^{\frac{3\pi i \nu_2^2 w_2}{2}}. \end{aligned}$$

Summing up gives

$$\begin{aligned} \sum_{\alpha \in \mathcal{S}^*} \varepsilon(\alpha) \theta_2(\alpha; w_1, w_2) &= \frac{1}{p^2} \sum_{B \in \mathcal{B}} \varepsilon_2(B) \sum_{\nu_1 \equiv B_1 \pmod{2p}} \nu_1 e^{\frac{\pi i \nu_1^2 w_1}{2p^2}} \sum_{\nu_2 \equiv B_2 \pmod{2p}} \nu_2 e^{\frac{3\pi i \nu_2^2 w_2}{2p^2}} \\ &= \frac{1}{p^2} \sum_{B \in \mathcal{B}} \varepsilon_2(B) \Theta_1\left(2p, B_1, 2p; \frac{w_1}{p}\right) \Theta_1\left(2p, B_2, 2p; \frac{3w_2}{p}\right) \end{aligned}$$

with

$$\mathcal{B} := \{(p+1, p-1), (1, -1), (p+2, p), (2, 0), (1, 1), (p+1, p+1)\}, \varepsilon_2(B) := \varepsilon\left(B_1, \frac{B_2 - 3B_1}{2}\right).$$

Combining the above yields that

$$\mathcal{E}_1(\tau) = -\frac{\sqrt{3}}{4p} \sum_{A \in \mathcal{A}} \varepsilon_1(A) \int_{-\tau}^{i\infty} \int_{w_1}^{i\infty} \frac{\Theta_1(2p, A_1, 2p; 3w_1) \Theta_1(2p, A_2, 2p; w_2)}{\sqrt{-i(w_1 + \tau)} \sqrt{-i(w_2 + \tau)}} dw_2 dw_1$$



$$-\frac{\sqrt{3}}{4p} \sum_{B \in \mathcal{B}} \varepsilon_2(B) \int_{-\bar{\tau}}^{i\infty} \int_{w_1}^{i\infty} \frac{\Theta_1(2p, B_1, 2p; w_1) \Theta_1(2p, B_2, 2p; 3w_2)}{\sqrt{-i(w_1 + \tau)} \sqrt{-i(w_2 + \tau)}} dw_2 dw_1.$$

Then, for  $M \in \Gamma_p$ , we have, using (2.9) and (2.10),

$$\Theta_1(2p, A, 2p; \ell M \tau) = \pm \left( \frac{\ell p c}{d} \right) \varepsilon_d(c\tau + d)^{\frac{3}{2}} \Theta_1(2p, A, 2p; \ell \tau).$$

Theorem 5.1 then finishes the claim using that  $\varepsilon_d^2 = (\frac{-1}{d})$ .

□

**5.3. Special multiple Eichler integrals of weight two.** Define for  $\alpha \in \mathcal{S}^*$

$$\begin{aligned} \mathcal{E}_{2,\alpha}(\tau) := & \frac{\sqrt{3}}{8\pi} \int_{-\bar{\tau}}^{i\infty} \int_{w_1}^{i\infty} \frac{2\theta_3(\alpha; w_1, w_2) - \theta_4(\alpha; w_1, w_2)}{\sqrt{-i(w_1 + \tau)} (-i(w_2 + \tau))^{\frac{3}{2}}} dw_2 dw_1 \\ & + \frac{\sqrt{3}}{8\pi} \int_{-\bar{\tau}}^{i\infty} \int_{w_1}^{i\infty} \frac{\theta_5(\alpha; w_1, w_2)}{(-i(w_1 + \tau))^{\frac{3}{2}} \sqrt{-i(w_2 + \tau)}} dw_2 dw_1 \end{aligned}$$

with

$$\begin{aligned} \theta_3(\alpha; w_1, w_2) &:= \sum_{n \in \alpha + \mathbb{Z}^2} (2n_1 + n_2) e^{\frac{3\pi i}{2}(2n_1 + n_2)^2 w_1 + \frac{\pi i n_2^2 w_2}{2}}, \\ \theta_4(\alpha; w_1, w_2) &:= \sum_{n \in \alpha + \mathbb{Z}^2} (3n_1 + 2n_2) e^{\frac{\pi i}{2}(3n_1 + 2n_2)^2 w_1 + \frac{3\pi i n_2^2 w_2}{2}}, \\ \theta_5(\alpha; w_1, w_2) &:= \sum_{n \in \alpha + \mathbb{Z}^2} n_1 e^{\frac{\pi i}{2}(3n_1 + 2n_2)^2 w_1 + \frac{3\pi i n_2^2 w_2}{2}}. \end{aligned}$$

We then set

$$\mathcal{E}_2(\tau) := \sum_{\alpha \in \mathcal{S}^*} \mathcal{E}_{2,\alpha}(p\tau).$$

This function again transforms quantum modular.

**Proposition 5.3.** *We have, for  $M \in \Gamma_p$ ,*

$$\mathcal{E}_2(\tau) - \left( \frac{3}{d} \right) (c\tau + d)^{-2} \mathcal{E}_2(M\tau) = \sum_{j=1}^{18} \left( r_{f_j, g_j, \frac{d}{c}}(\tau) + I_{f_j}(\tau) r_{g_j, \frac{d}{c}}(\tau) \right),$$

where  $f_j$  and  $g_j$  are holomorphic modular forms of weight  $\frac{1}{2}$  or cusp forms of weight  $\frac{3}{2}$ .

*Proof.* As in the proof of Proposition 5.2, we obtain

$$\begin{aligned} \sum_{\substack{\alpha \in \mathcal{S}^* \\ n \in \alpha + \mathbb{Z}^2}} (2n_1 + n_2) e^{\frac{3\pi i}{2}(2n_1 + n_2)^2 w_1 + \frac{\pi i n_2^2 w_2}{2}} &= \frac{1}{p} \sum_{A \in \mathcal{A}} \Theta_1 \left( 2p, A_1, 2p; \frac{3w_1}{p} \right) \Theta_0 \left( 2p, A_2, 2p; \frac{w_2}{p} \right), \\ \sum_{\substack{\alpha \in \mathcal{S}^* \\ n \in \alpha + \mathbb{Z}^2}} (3n_1 + 2n_2) e^{\frac{\pi i}{2}(3n_1 + 2n_2)^2 w_1 + \frac{3\pi i n_2^2 w_2}{2}} &= \frac{1}{p} \sum_{B \in \mathcal{B}} \Theta_1 \left( 2p, B_1, 2p; \frac{w_1}{p} \right) \Theta_0 \left( 2p, B_2, 2p; \frac{3w_2}{p} \right), \end{aligned}$$

$$\sum_{\substack{\alpha \in \mathcal{S}^* \\ n \in \alpha + \mathbb{Z}^2}} n_1 e^{\frac{\pi i}{2}(3n_1 + 2n_2)^2 w_1 + \frac{3\pi i}{2} n_1^2 w_2} = \frac{1}{p} \sum_{B \in \mathcal{B}} \Theta_0 \left( 2p, B_1, 2p; \frac{w_1}{p} \right) \Theta_1 \left( 2p, B_2, 2p; \frac{3w_2}{p} \right).$$

The claim now again follows from Theorem 5.1 using (2.9) and (2.10).  $\square$

**5.4. More on double Eichler integrals.** We have an obvious map  $S_k(\Gamma, \chi) \rightarrow \mathcal{Q}_{2-k}(\Gamma^*, \chi^*)$ , where  $\chi^*(M) := \chi(M^*)$ , which assigns to  $f \in S_k(\Gamma, \chi)$  its Eichler integral  $I_f$ , defined in (2.11). Clearly, we also have a map from  $S_k(\Gamma, \chi) \otimes S_k(\Gamma, \chi)$ , actually from its symmetric square, to  $(\mathcal{Q}_{2-k}(\Gamma^*, \chi^*))^2$ , by mapping  $f_1 \otimes f_2$  to  $I_{f_1} I_{f_2}$ . The double Eichler integral construction  $I_{f_1, f_2}$  gives rise to a map

$$\Lambda^2(S_k(\Gamma, \chi)) \longrightarrow \mathcal{Q}_{4-2k}^2(\Gamma^*, \chi^{*2}) / (\mathcal{Q}_{2-k}(\Gamma^*, \chi^*))^2,$$

where  $\Lambda^2(S_{2-k}(\Gamma, \chi))$  is the second exterior power of  $S_{2-k}(\Gamma, \chi)$ . To see this, it suffices to observe the simplest *shuffle* relation for iterated integrals

$$I_{f_1, f_2} + I_{f_2, f_1} = I_{f_1} I_{f_2}.$$

This relation follows directly from

$$\begin{aligned} & \int_{-\bar{\tau}}^{i\infty} \int_{w_1}^{i\infty} \frac{f_1(w_1) f_2(w_2)}{(-i(w_1 + \tau))^{2-k_1} (-i(w_2 + \tau))^{2-k_2}} dw_2 dw_1 \\ &= \int_{-\bar{\tau}}^{i\infty} \int_{-\bar{\tau}}^{w_2} \frac{f_1(w_1) f_2(w_2)}{(-i(w_1 + \tau))^{2-k_1} (-i(w_2 + \tau))^{2-k_2}} dw_1 dw_2 \\ &= \int_{-\bar{\tau}}^{i\infty} \int_{-\bar{\tau}}^{i\infty} \frac{f_1(w_1) f_2(w_2)}{(-i(w_1 + \tau))^{2-k_1} (-i(w_2 + \tau))^{2-k_2}} dw_1 dw_2 \\ &\quad - \int_{-\bar{\tau}}^{i\infty} \int_{w_2}^{i\infty} \frac{f_1(w_1) f_2(w_2)}{(-i(w_1 + \tau))^{2-k_1} (-i(w_2 + \tau))^{2-k_2}} dw_1 dw_2. \end{aligned}$$

*Remark.* It is now straightforward to consider even more general iterated Eichler integrals ( $r \in \mathbb{N}$ ):

$$I_{f_1, \dots, f_r} := \int_{-\bar{\tau}}^{i\infty} \int_{w_{r-1}}^{i\infty} \cdots \int_{w_2}^{i\infty} \prod_{j=1}^r \frac{f_j(w_j)}{(-i(w_j + \tau))^{2-k_j}} dw_1 \cdots dw_r,$$

where the  $f_j$  are cusp forms of weight  $k_j \geq \frac{1}{2}$  (or possibly holomorphic forms for weight  $\frac{1}{2}$ ). We do not pursue their (mock/quantum) modular properties here - we will address this in our future work (see also Section 9 for related comments).

## 6. INDEFINITE THETA FUNCTIONS

We next realize the double Eichler integrals as pieces of indefinite theta functions.

**6.1. The function  $\mathcal{E}_1$  as indefinite theta function.** The next lemma rewrites  $\mathbb{E}_1(\tau) := \mathcal{E}_1(\frac{\tau}{p})$  in a shape to which one can apply the Euler-Maclaurin summation formula.

**Lemma 6.1.** *We have*

$$\mathbb{E}_1(\tau) = \frac{1}{2} \sum_{\alpha \in \mathcal{S}^*} \varepsilon(\alpha) \sum_{n \in \alpha + \mathbb{Z}^2} M_2 \left( \sqrt{3}; \sqrt{v} \left( 2\sqrt{3}n_1 + \sqrt{3}n_2, n_2 \right) \right) q^{-Q(n)}.$$

*Proof.* The claim follows, once we prove that

$$\begin{aligned} & M_2 \left( \sqrt{3}; \sqrt{3v}(2n_1 + n_2), \sqrt{v}n_2 \right) \\ &= -\frac{\sqrt{3}}{2} (2n_1 + n_2) n_2 q^{Q(n_1, n_2)} \int_{-\tau}^{i\infty} \frac{e^{\frac{3\pi i}{2}(2n_1 + n_2)^2 w_1}}{\sqrt{-i(w_1 + \tau)}} \int_{w_1}^{i\infty} \frac{e^{\frac{\pi i n_2^2 w_2}{2}}}{\sqrt{-i(w_2 + \tau)}} dw_2 dw_1 \\ &\quad - \frac{\sqrt{3}}{2} (3n_1 + 2n_2) n_1 q^{Q(n_1, n_2)} \int_{-\tau}^{i\infty} \frac{e^{\frac{\pi i}{2}(3n_1 + 2n_2)^2 w_1}}{\sqrt{-i(w_1 + \tau)}} \int_{w_1}^{i\infty} \frac{e^{\frac{3\pi i n_1^2 w_2}{2}}}{\sqrt{-i(w_2 + \tau)}} dw_2 dw_1. \end{aligned} \quad (6.1)$$

For simplicity we only show (6.1) for  $n_1 \neq 0$ . Since, by (2.7),

$$\lim_{\lambda \rightarrow \infty} M_2(\kappa; \lambda u_1, \lambda u_2) = 0,$$

we obtain, using (2.5) and (2.6),

$$\begin{aligned} M_2(\kappa; u_1, u_2) &= - \int_1^\infty \frac{\partial}{\partial w_1} M_2(\kappa; w_1 u_1, w_1 u_2) dw_1 \\ &= - \int_1^\infty \left( u_1 M_2^{(1,0)}(\kappa; w_1 u_1, w_1 u_2) + u_2 M_2^{(0,1)}(\kappa; w_1 u_1, w_1 u_2) \right) dw_1 \\ &= -2 \int_1^\infty \left( u_1 e^{-\pi u_1^2 w_1^2} M(u_2 w_1) + \frac{u_2 + \kappa u_1}{\sqrt{1 + \kappa^2}} e^{-\frac{\pi(u_2 + \kappa u_1)^2 w_1^2}{1 + \kappa^2}} M\left(w_1 \frac{u_1 - \kappa u_2}{\sqrt{1 + \kappa^2}}\right) \right) dw_1 \\ &= - \int_1^\infty \left( u_1 e^{-\pi u_1^2 w_1} M(u_2 \sqrt{w_1}) + \frac{u_2 + \kappa u_1}{\sqrt{1 + \kappa^2}} e^{-\frac{\pi(u_2 + \kappa u_1)^2 w_1}{1 + \kappa^2}} M\left(\sqrt{w_1} \frac{u_1 - \kappa u_2}{\sqrt{1 + \kappa^2}}\right) \right) \frac{dw_1}{\sqrt{w_1}} \\ &= \frac{i}{\sqrt{2}} \int_{-\tau}^{i\infty} \left( \frac{u_1}{\sqrt{v}} e^{\frac{\pi i u_1^2 w_1}{2v}} q^{\frac{u_1^2}{4v}} M\left(\sqrt{\frac{-i(w_1 + \tau)}{2v}} u_2\right) \right. \\ &\quad \left. + \frac{u_2 + \kappa u_1}{\sqrt{(1 + \kappa^2)v}} e^{\frac{\pi i(u_2 + \kappa u_1)^2 w_1}{2(1 + \kappa^2)v}} q^{\frac{(u_2 + \kappa u_1)^2}{4(1 + \kappa^2)v}} M\left(\sqrt{\frac{-i(w_1 + \tau)}{2}} \frac{u_1 - \kappa u_2}{\sqrt{(1 + \kappa^2)v}}\right) \right) \frac{dw_1}{\sqrt{-i(w_1 + \tau)}}. \end{aligned} \quad (6.2)$$

Now write for  $N \in \mathbb{R}^+$

$$M\left(\sqrt{\frac{-i(w_1 + \tau)}{2}} N\right) = \frac{iN}{\sqrt{2}} q^{\frac{N^2}{4}} \int_{w_1}^{i\infty} \frac{e^{\frac{\pi i N^2 w_2}{2}}}{\sqrt{-i(w_2 + \tau)}} dw_2.$$

Plugging this into (6.2) easily yields that

$$M_2(\kappa; u_1, u_2) = -\frac{u_1}{2\sqrt{v}} \frac{u_2}{\sqrt{v}} q^{\frac{u_1^2}{4v} + \frac{u_2^2}{4v}} \int_{-\tau}^{i\infty} \frac{e^{\frac{\pi i u_1^2 w_1}{2v}}}{\sqrt{-i(w_1 + \tau)}} \int_{w_1}^{i\infty} \frac{e^{\frac{\pi i u_2^2 w_2}{2v}}}{\sqrt{-i(w_2 + \tau)}} dw_2 dw_1$$

$$-\frac{u_2 + \kappa u_1}{2\sqrt{(1+\kappa^2)v}} \frac{u_1 - \kappa u_2}{\sqrt{(1+\kappa^2)v}} q^{\frac{(u_2 + \kappa u_1)^2}{4(1+\kappa^2)v} + \frac{(u_1 - \kappa u_2)^2}{4(1+\kappa^2)v}} \int_{-\tau}^{i\infty} \frac{e^{\frac{\pi i(u_2 + \kappa u_1)^2 w_1}{2(1+\kappa^2)v}}}{\sqrt{-i(w_1 + \tau)}} \int_{w_1}^{i\infty} \frac{e^{\frac{\pi i(u_1 - \kappa u_2)^2 w_2}{2(1+\kappa^2)v}}}{\sqrt{-i(w_2 + \tau)}} dw_2 dw_1.$$

From this it is not hard to conclude (6.1).  $\square$

**6.2. The function  $\mathcal{E}_2$  as indefinite theta function.** We next write  $\mathbb{E}_2(\tau) := \mathcal{E}_2(\frac{\tau}{p})$  as a piece of a derivative of an indefinite theta function, having an extra Jacobi variable.

**Lemma 6.2.** *We have*

$$\begin{aligned} \mathbb{E}_2(\tau) &= \frac{1}{4\pi i} \sum_{\alpha \in \mathcal{J}^*} \\ &\quad \times \sum_{n \in \alpha + \mathbb{Z}^2} \left[ \frac{\partial}{\partial z} \left( M_2 \left( \sqrt{3}; \sqrt{3v}(2n_1 + n_2), \sqrt{v} \left( n_2 - \frac{2\text{Im}(z)}{v} \right) \right) e^{2\pi i n_2 z} \right) \right]_{z=0} q^{-Q(n)}. \end{aligned}$$

*Proof.* We first compute

$$\begin{aligned} \frac{1}{2\pi i} \left[ \frac{\partial}{\partial z} \left( M_2 \left( \sqrt{3}; \sqrt{3v}(2n_1 + n_2), \sqrt{v} \left( n_2 - \frac{2\text{Im}(z)}{v} \right) \right) e^{2\pi i n_2 z} \right) \right]_{z=0} \\ = n_2 M_2 \left( \sqrt{3}; \sqrt{3v}(2n_1 + n_2), \sqrt{v} n_2 \right) + \frac{1}{2\pi \sqrt{v}} e^{-\pi(3n_1 + 2n_2)^2 v} M \left( \sqrt{3v} n_1 \right). \end{aligned} \quad (6.3)$$

We show below that

$$\begin{aligned} n_2 M_2 \left( \sqrt{3}; \sqrt{3v}(2n_1 + n_2), \sqrt{v} n_2 \right) &= -\frac{\sqrt{3}}{2\pi} (2n_1 + n_2) \int_{2v}^{\infty} \frac{e^{-\frac{3\pi}{2}(2n_1 + n_2)^2 w_1}}{\sqrt{w_1}} \int_{w_1}^{\infty} \frac{e^{-\frac{\pi n_2^2 w_2}{2}}}{w_2^{\frac{3}{2}}} dw_2 dw_1 \\ &+ \frac{\sqrt{3}}{4\pi} (3n_1 + 2n_2) \int_{2v}^{\infty} \frac{e^{-\frac{\pi}{2}(3n_1 + 2n_2)^2 w_1}}{\sqrt{w_1}} \int_{w_1}^{\infty} \frac{e^{-\frac{3\pi n_1^2 w_2}{2}}}{w_2^{\frac{3}{2}}} dw_2 dw_1 - \frac{1}{2\pi \sqrt{v}} e^{-\pi(3n_1 + 2n_2)^2 v} M \left( \sqrt{3v} n_1 \right) \\ &- \frac{\sqrt{3} n_1}{4\pi} \int_{2v}^{\infty} \frac{e^{-\frac{\pi}{2}(3n_1 + 2n_2)^2 w_1}}{w_1^{\frac{3}{2}}} \int_{w_1}^{\infty} \frac{e^{-\frac{3\pi n_1^2 w_2}{2}}}{w_2^{\frac{1}{2}}} dw_2 dw_1. \end{aligned} \quad (6.4)$$

Since the third term cancels the second term on the right-hand side of (6.3) this then implies the claim, using that

$$\begin{aligned} \int_{2v}^{\infty} \frac{e^{-2\pi M^2 w_1}}{w_1^{\frac{1}{2}}} \int_{w_1}^{\infty} \frac{e^{-2\pi N^2 w_2}}{w_2^{\frac{3}{2}}} dw_2 dw_1 &= -q^{M^2 + N^2} \int_{-\tau}^{i\infty} \frac{e^{2\pi i M^2 w_1}}{(-i(w_1 + \tau))^{\frac{1}{2}}} \int_{w_1}^{i\infty} \frac{e^{2\pi i N^2 w_2}}{(-i(w_2 + \tau))^{\frac{3}{2}}} dw_2 dw_1, \\ \int_{2v}^{\infty} \frac{e^{-2\pi M^2 w_1}}{w_1^{\frac{3}{2}}} \int_{w_1}^{\infty} \frac{e^{-2\pi N^2 w_2}}{w_2^{\frac{1}{2}}} dw_2 dw_1 &= -q^{N^2 + M^2} \int_{-\tau}^{i\infty} \frac{e^{2\pi i M^2 w_1}}{(-i(w_1 + \tau))^{\frac{3}{2}}} \int_{w_1}^{i\infty} \frac{e^{2\pi i N^2 w_2}}{(-i(w_2 + \tau))^{\frac{1}{2}}} dw_2 dw_1. \end{aligned}$$

To prove (6.4), we again, for simplicity, restrict to  $n_1 \neq 0$ . Plugging in (6.2) yields

$$M_2 \left( \sqrt{3}; \sqrt{3v}(2n_1 + n_2), \sqrt{v}n_2 \right) = - \int_1^\infty \left( \sqrt{3v}(2n_1 + n_2) e^{-3\pi v(2n_1 + n_2)^2 w_1} M(\sqrt{vw_1}n_2) \right. \\ \left. + \sqrt{v}(3n_1 + 2n_2) e^{-\pi v(3n_1 + 2n_2)^2 w_1} M(\sqrt{3vw_1}n_1) \right) \frac{dw_1}{\sqrt{w_1}}. \quad (6.5)$$

Using (2.3) and (2.2) the first term in (6.5) multiplied by  $n_2$  gives

$$- \frac{\sqrt{3v}}{2\sqrt{\pi}} |n_2| (2n_1 + n_2) \int_1^\infty e^{-3\pi v(2n_1 + n_2)^2 w_1} \Gamma \left( -\frac{1}{2}, \pi v n_2^2 w_1 \right) \frac{dw_1}{\sqrt{w_1}} \\ + \frac{\sqrt{3}}{\pi} (2n_1 + n_2) \int_1^\infty e^{-4\pi v Q(n_1, n_2) w_1} \frac{dw_1}{w_1}. \quad (6.6)$$

For the second term in (6.5), we split

$$n_2 = \frac{1}{2}(3n_1 + 2n_2) - \frac{3}{2}n_1. \quad (6.7)$$

The  $n_1$ -term contributes to  $n_2 M_2$  as

$$\frac{3}{4} \sqrt{\frac{v}{\pi}} |n_1| (3n_1 + 2n_2) \int_1^\infty e^{-\pi v(3n_1 + 2n_2)^2 w_1} \Gamma \left( -\frac{1}{2}, 3\pi v n_1^2 w_1 \right) \frac{dw_1}{\sqrt{w_1}} \\ - \frac{\sqrt{3}}{2\pi} (3n_1 + 2n_2) \int_1^\infty e^{-4\pi v Q(n_1, n_2) w_1} \frac{dw_1}{w_1}. \quad (6.8)$$

We next use that for  $N \in \mathbb{N}_0$ ,  $M \in \mathbb{N}$

$$\int_1^\infty e^{-4\pi N^2 v w_1} \Gamma \left( -\frac{1}{2}, 4\pi v M^2 w_1 \right) \frac{dw_1}{\sqrt{w_1}} = \frac{1}{2\sqrt{\pi v} |M|} \int_{2v}^\infty \frac{e^{-2\pi N^2 w_1}}{\sqrt{w_1}} \int_{w_1}^\infty \frac{e^{-2\pi M^2 w_2}}{w_2^{\frac{3}{2}}} dw_2 dw_1. \quad (6.9)$$

We use this to rewrite the first terms in (6.6) and (6.8). The first term in (6.6) is the first term on the right-hand side of (6.4). Similarly, since  $n_1 \neq 0$ , the first term in (6.8) equals the second term in (6.4). Now we combine the second terms in (6.6) and (6.8), to get

$$\frac{\sqrt{3}n_1}{2\pi} \int_1^\infty e^{-4\pi v Q(n_1, n_2) w_1} \frac{dw_1}{w_1}. \quad (6.10)$$

Next compute the contribution from the first term in (6.7),

$$- \frac{\sqrt{v}}{2} (3n_1 + 2n_2)^2 \int_1^\infty e^{-\pi v(3n_1 + 2n_2)^2 w_1} M(\sqrt{3vw_1}n_1) \frac{dw_1}{\sqrt{w_1}} \\ = \frac{1}{2\pi\sqrt{v}} \int_1^\infty \frac{\partial}{\partial w_1} \left( e^{-\pi v(3n_1 + 2n_2)^2 w_1} \right) \frac{M(\sqrt{3vw_1}n_1)}{\sqrt{w_1}} dw_1.$$

Using integration by parts, this becomes

$$- \frac{1}{2\pi\sqrt{v}} e^{-\pi v(3n_1 + 2n_2)^2} M(\sqrt{3v}n_1) - \frac{\sqrt{3}n_1}{2\pi} \int_1^\infty e^{-4\pi v Q(n_1, n_2) w_1} \frac{dw_1}{w_1}$$

$$+ \frac{1}{4\pi\sqrt{v}} \int_1^\infty e^{-\pi v(3n_1+2n_2)^2 w_1} \frac{M(\sqrt{3vw_1}n_1)}{w_1^{\frac{3}{2}}} dw_1. \quad (6.11)$$

The second term now cancels (6.10) and the first term equals the third term in (6.4).

To rewrite the final term in (6.11), we use, that for  $M, N \in \mathbb{Z}$  with  $N \neq 0$

$$\int_1^\infty e^{-4\pi v M^2 w_1} \frac{M(2\sqrt{vw_1}N)}{w_1^{\frac{3}{2}}} dw_1 = -2\sqrt{v}N \int_{2v}^\infty \frac{e^{-2\pi M^2 w_1}}{w_1^{\frac{3}{2}}} \int_{w_1}^\infty \frac{e^{-2\pi N^2 w_2}}{w_2^{\frac{1}{2}}} dw_2 dw_1.$$

Thus the last term in (6.11) gives the final term in (6.4).  $\square$

## 7. ASYMPTOTIC BEHAVIOR OF MULTIPLE EICHLER INTEGRALS AND PROOF OF THEOREM 1.1

In this section, we asymptotically relate  $F_j$  and  $\mathbb{E}_j$ .

### 7.1. Asymptotic behavior of $\mathbb{E}_1$ . Write

$$F_1\left(e^{2\pi i \frac{h}{k} - t}\right) \sim \sum_{m \geq 0} a_{h,k}(m) t^m \quad (t \rightarrow 0^+).$$

The goal of this section is to prove the following

**Theorem 7.1.** *We have, for  $h, k \in \mathbb{Z}$  with  $k > 0$  and  $\gcd(h, k) = 1$ ,*

$$\mathbb{E}_1\left(\frac{h}{k} + \frac{it}{2\pi}\right) \sim \sum_{m \geq 0} a_{-h,k}(m) (-t)^m \quad (t \rightarrow 0^+).$$

*Proof.* We use Lemma 6.1 and the fact that  $M_2$  is an even function, to rewrite

$$\begin{aligned} \mathbb{E}_1(\tau) &= \frac{1}{2} \sum_{\alpha \in \mathcal{S}} \varepsilon(\alpha) \sum_{n \in \alpha + \mathbb{N}_0^2} M_2\left(\sqrt{3}; \sqrt{v} \left(2\sqrt{3}n_1 + \sqrt{3}n_2, n_2\right)\right) q^{-Q(n_1, n_2)} \\ &\quad + \frac{1}{2} \sum_{\alpha \in \tilde{\mathcal{S}}} \tilde{\varepsilon}(\alpha) \sum_{n \in \alpha + \mathbb{N}_0^2} M_2\left(\sqrt{3}; \sqrt{v} \left(-2\sqrt{3}n_1 + \sqrt{3}n_2, n_2\right)\right) q^{-Q(-n_1, n_2)}, \end{aligned}$$

where

$$\tilde{\mathcal{S}} := \{(1 - \alpha_1, \alpha_2) : \alpha \in \mathcal{S}\}, \quad \tilde{\varepsilon}(\alpha_1, \alpha_2) := \varepsilon(1 - \alpha_1, \alpha_2).$$

To apply the Euler-Maclaurin summation formula directly, we turn every  $\text{sgn}$  into  $\text{sgn}^*$ , where  $\text{sgn}^*(x) := \text{sgn}(x)$  for  $x \neq 0$  and  $\text{sgn}^*(0) := 1$ . To be more precise, we set

$$\begin{aligned} M_2^*\left(\sqrt{3}; \sqrt{3}(2x_1 + x_2), x_2\right) &:= \text{sgn}^*(x_1) \text{sgn}^*(x_2) + E_2\left(\sqrt{3}; \sqrt{3}(2x_1 + x_2), x_2\right) \\ &\quad - \text{sgn}^*(x_2) E\left(\sqrt{3}(2x_1 + x_2)\right) - \text{sgn}^*(x_1) E(3x_1 + 2x_2). \end{aligned} \quad (7.1)$$

Using that

$$M_2\left(\sqrt{3}; \sqrt{3}x_2, x_2\right) - \lim_{x_1 \rightarrow 0^+} M_2^*\left(\sqrt{3}; \sqrt{3}(\pm 2x_1 + x_2), x_2\right) = \pm M(2x_2), \quad (7.2)$$

we then split

$$\mathbb{E}_1(\tau) = \mathcal{E}_1^*(\tau) + H_1(\tau)$$

with

$$\begin{aligned}\mathcal{E}_1^*(\tau) &:= \frac{1}{2} \sum_{\alpha \in \mathcal{S}} \varepsilon(\alpha) \sum_{n \in \alpha + \mathbb{N}_0^2} M_2^* \left( \sqrt{3}; \sqrt{v} \left( 2\sqrt{3}n_1 + \sqrt{3}n_2, n_2 \right) \right) q^{-Q(n_1, n_2)} \\ &\quad + \frac{1}{2} \sum_{\alpha \in \tilde{\mathcal{S}}} \tilde{\varepsilon}(\alpha) \sum_{n \in \alpha + \mathbb{N}_0^2} M_2^* \left( \sqrt{3}; \sqrt{v} \left( -2\sqrt{3}n_1 + \sqrt{3}n_2, n_2 \right) \right) q^{-Q(-n_1, n_2)}, \\ H_1(\tau) &:= -\frac{1}{2} \sum_{m \in \frac{1}{p} + \mathbb{N}_0} M(2\sqrt{v}m) q^{-m^2} + \frac{1}{2} \sum_{m \in 1 - \frac{1}{p} + \mathbb{N}_0} M(2\sqrt{v}m) q^{-m^2}.\end{aligned}$$

Note that for  $n_1 = 0$  we take the limit  $n_1 \rightarrow 0$  in the  $M_2^*$ -functions.

We proceed as in Subsection 4.1 to determine the asymptotic behavior of  $\mathcal{E}_1^*$  and  $H_1$ . Firstly we rewrite

$$\begin{aligned}\mathcal{E}_1^* \left( \frac{h}{k} + \frac{it}{2\pi} \right) &= \sum_{\alpha \in \mathcal{S}} \varepsilon(\alpha) \sum_{0 \leq \ell \leq \frac{kp}{\delta} - 1} e^{-2\pi i \frac{h}{k} Q(\ell_1 + \alpha_1, \ell_2 + \alpha_2)} \sum_{n \in \frac{\delta(\ell + \alpha)}{kp} + \mathbb{N}_0^2} \mathcal{F}_3 \left( \frac{kp}{\delta} \sqrt{tn} \right) \\ &\quad + \sum_{\alpha \in \tilde{\mathcal{S}}} \tilde{\varepsilon}(\alpha) \sum_{0 \leq \ell \leq \frac{kp}{\delta} - 1} e^{-2\pi i \frac{h}{k} Q(-(\ell_1 + \alpha_1), \ell_2 + \alpha_2)} \sum_{n \in \frac{\delta(\ell + \alpha)}{kp} + \mathbb{N}_0^2} \tilde{\mathcal{F}}_3 \left( \frac{kp}{\delta} \sqrt{tn} \right),\end{aligned}$$

where

$$\mathcal{F}_3(x_1, x_2) := \frac{1}{2} M_2^* \left( \sqrt{3}; \frac{1}{\sqrt{2\pi}} \left( \sqrt{3}(2x_1 + x_2), x_2 \right) \right) e^{Q(x)}, \quad \tilde{\mathcal{F}}_3(x_1, x_2) := \mathcal{F}_3(-x_1, x_2).$$

The contribution from the  $\mathcal{F}_3$  term to the first term in (2.8) is

$$\frac{\delta^2}{k^2 p^2 t} \mathcal{I}_{\mathcal{F}_3} \sum_{\alpha \in \mathcal{S}} \varepsilon(\alpha) \sum_{0 \leq \ell \leq \frac{kp}{\delta} - 1} e^{-2\pi i \frac{h}{k} Q(\ell + \alpha)} = 0,$$

conjugating (4.2). In the same way the main term coming from  $\tilde{\mathcal{F}}_3$  is shown to vanish.

The contribution to the second term of Euler-Maclaurin is

$$\begin{aligned}-2 \sum_{\alpha \in \mathcal{S}^*} \varepsilon(\alpha) \sum_{0 \leq \ell \leq \frac{kp}{\delta} - 1} e^{-2\pi i \frac{h}{k} Q(\ell + \alpha)} \sum_{n_2 \geq 0} \frac{B_{2n_2+2} \left( \frac{\delta(\ell_2 + \alpha_2)}{kp} \right)}{(2n_2 + 2)!} \\ \times \int_0^\infty \left( \mathcal{F}_3^{(0, 2n_2+1)}(x_1, 0) + \tilde{\mathcal{F}}_3^{(0, 2n_2+1)}(x_1, 0) \right) dx_1 \left( \frac{k^2 p^2 t}{\delta^2} \right)^{n_2}.\end{aligned}$$

We now claim that

$$\int_0^\infty \left( \mathcal{F}_3^{(0, 2n_2+1)}(x_1, 0) + \tilde{\mathcal{F}}_3^{(0, 2n_2+1)}(x_1, 0) \right) dx_1 = (-1)^{n_2} \int_0^\infty \mathcal{F}_1^{(0, 2n_2+1)}(x_1, 0) dx_1. \quad (7.3)$$

Firstly the right-hand side of (7.3) equals

$$\left[ \frac{\partial^{2n_2+1}}{\partial x_2^{2n_2+1}} \int_0^\infty \mathcal{F}_1(x_1, x_2) dx_1 \right]_{x_2=0} = \left[ \frac{\partial^{2n_2+1}}{\partial x_2^{2n_2+1}} \left( e^{-\frac{x_2^2}{4}} \int_0^\infty e^{-3(x_1 + \frac{x_2}{2})^2} dx_1 \right) \right]_{x_2=0}. \quad (7.4)$$

Now the integral in (7.4) evaluates as

$$\sqrt{\frac{\pi}{3}} \int_{\frac{\sqrt{3}x_2}{2\sqrt{\pi}}}^{\infty} e^{-\pi x_1^2} dx_1 = \frac{\sqrt{\pi}}{2\sqrt{3}} \left( 1 - E \left( \frac{\sqrt{3}x_2}{2\sqrt{\pi}} \right) \right).$$

Thus (7.4) becomes

$$\frac{\sqrt{\pi}}{2\sqrt{3}} \left[ \frac{\partial^{2n_2+1}}{\partial x_2^{2n_2+1}} \left( e^{-\frac{x_2^2}{4}} \left( 1 - E \left( \frac{\sqrt{3}x_2}{2\sqrt{\pi}} \right) \right) \right) \right]_{x_2=0} = -\frac{\sqrt{\pi}}{2\sqrt{3}} \left[ \frac{\partial^{2n_2+1}}{\partial x_2^{2n_2+1}} \left( e^{-\frac{x_2^2}{4}} E \left( \frac{\sqrt{3}x_2}{2\sqrt{\pi}} \right) \right) \right]_{x_2=0}. \quad (7.5)$$

To compute the left-hand side of (7.3), we decompose, according to (7.1),

$$\begin{aligned} M_2^* \left( \sqrt{3}, \sqrt{3}(2x_1 + x_2), x_2 \right) \\ = \operatorname{sgn}^*(x_1) \operatorname{sgn}^*(x_2) + h_1(x_1, x_2) - \operatorname{sgn}^*(x_2) h_2(x_1, x_2) - \operatorname{sgn}^*(x_1) h_3(x_1, x_2), \end{aligned}$$

where

$$\begin{aligned} h_1(x_1, x_2) &:= E_2 \left( \sqrt{3}; \sqrt{3}(2x_1 + x_2), x_2 \right), \\ h_2(x_1, x_2) &:= E \left( \sqrt{3}(2x_1 + x_2) \right), \quad h_3(x_1, x_2) := E(3x_1 + 2x_2). \end{aligned}$$

Setting

$$a_0(x_1, x_2) := e^{Q(x_1, x_2)}, \quad a_j(x_1, x_2) := h_j \left( \frac{1}{\sqrt{2\pi}}(x_1, x_2) \right) e^{Q(x_1, x_2)},$$

we then obtain

$$\begin{aligned} \mathcal{F}_3^{(0, 2n_2+1)}(x_1, 0) + \tilde{\mathcal{F}}_3^{(0, 2n_2+1)}(x_1, 0) \\ = \frac{1}{2} \left( a_0^{(0, 2n_2+1)}(x_1, 0) + a_1^{(0, 2n_2+1)}(x_1, 0) - a_2^{(0, 2n_2+1)}(x_1, 0) - a_3^{(0, 2n_2+1)}(x_1, 0) \right) \\ + \frac{1}{2} \left( -a_0^{(0, 2n_2+1)}(-x_1, 0) + a_1^{(0, 2n_2+1)}(-x_1, 0) - a_2^{(0, 2n_2+1)}(-x_1, 0) + a_3^{(0, 2n_2+1)}(-x_1, 0) \right) \quad (7.6) \\ = a_0^{(0, 2n_2+1)}(x_1, 0) - a_2^{(0, 2n_2+1)}(x_1, 0), \quad (7.7) \end{aligned}$$

using that  $a_0$  and  $a_1$  are even and  $a_2$  and  $a_3$  are odd. Plugging in the definition of  $a_0$  and  $a_2$ , we need to consider

$$-\left[ \frac{\partial^{2n_2+1}}{\partial x_2^{2n_2+1}} \left( e^{\frac{x_2^2}{2}} \int_0^{\infty} e^{3x_1^2 + 3x_1 x_2} M \left( \sqrt{\frac{3}{2\pi}}(2x_1 + x_2) \right) dx_1 \right) \right]_{x_2=0}. \quad (7.8)$$

Changing variables  $w := \sqrt{\frac{3}{2\pi}}(2x_1 + x_2)$ , the function in (7.8) before differentiation is

$$-\sqrt{\frac{\pi}{6}} e^{\frac{x_2^2}{4}} \int_{\sqrt{\frac{3}{2\pi}}x_2}^{\infty} M(w) e^{\frac{\pi w^2}{2}} dw = -\sqrt{\frac{\pi}{6}} e^{\frac{x_2^2}{4}} \left( \int_0^{\infty} M(w) e^{\frac{\pi w^2}{2}} dw - \int_0^{\sqrt{\frac{3}{2\pi}}x_2} M(w) e^{\frac{\pi w^2}{2}} dw \right).$$

The first integral vanishes upon differentiating an odd number of times and then setting  $x_2 = 0$ . In the second integral we decompose  $M(w) = E(w) - 1$ . The contribution of the  $E$ -function vanishes,



since  $E$  is an odd function. We are left with

$$-\sqrt{\frac{\pi}{6}} \left[ \frac{\partial^{2n_2+1}}{\partial x_2^{2n_2+1}} \left( e^{\frac{x_2^2}{4}} \int_0^{\sqrt{\frac{3}{2\pi}} x_2} e^{\frac{\pi w^2}{2}} dw \right) \right]_{x_2=0} = -\sqrt{\frac{\pi}{6}} i^{-2n_2-1} \left[ \frac{\partial^{2n_2+1}}{\partial x_2^{2n_2+1}} \left( e^{-\frac{x_2^2}{4}} \int_0^{\sqrt{\frac{3}{2\pi}} x_2 i} e^{\frac{\pi w^2}{2}} dw \right) \right]_{x_2=0}.$$

The integral now is

$$i\sqrt{2} \int_0^{\frac{\sqrt{3}x_2}{2\sqrt{\pi}}} e^{-\pi w^2} dw = \frac{i}{\sqrt{2}} E \left( \frac{\sqrt{3}x_2}{2\sqrt{\pi}} \right).$$

Thus we obtain

$$\frac{\sqrt{\pi}}{2\sqrt{3}} (-1)^{n_2+1} \left[ \frac{\partial^{2n_2+1}}{\partial x_2^{2n_2+1}} \left( e^{-\frac{x_2^2}{4}} E \left( \frac{\sqrt{3}x_2}{2\sqrt{\pi}} \right) \right) \right]_{x_2=0},$$

as claimed, by comparing with (7.5).

In the same way one can show that the third term in Euler-Maclaurin equals

$$-2 \sum_{\alpha \in \mathcal{S}^*} \varepsilon(\alpha) \sum_{0 \leq \ell \leq \frac{kp}{\delta} - 1} e^{-2\pi i \frac{h}{k} Q(\ell+\alpha)} \sum_{n_2 \geq 0} \frac{B_{2n_2+2} \left( \frac{\delta(\ell_1+\alpha_1)}{kp} \right)}{(2n_2+2)!} \int_0^\infty \mathcal{F}_1^{(2n_2+1,0)}(0, x_2) dx_2 \left( \frac{k^2 p^2 t}{\delta^2} \right)^{n_2}.$$

The contribution to the final term is, pairing as in Section 4

$$2 \sum_{\alpha \in \mathcal{S}^*} \varepsilon(\alpha) \sum_{0 \leq \ell \leq \frac{kp}{\delta} - 1} e^{-2\pi i \frac{h}{k} Q(\ell+\alpha)} \sum_{\substack{n_1, n_2 \geq 0 \\ n_1 \equiv n_2 \pmod{2}}} \frac{B_{n_1+1} \left( \frac{\delta(\ell_1+\alpha_1)}{kp} \right)}{(n_1+1)!} \frac{B_{n_2+1} \left( \frac{\delta(\ell_2+\alpha_2)}{kp} \right)}{(n_1+1)!} \\ \times \left( \mathcal{F}_3^{(n_1, n_2)}(0, 0) - (-1)^{n_1} \tilde{\mathcal{F}}_3^{(n_1, n_2)}(0, 0) \right) \left( \frac{kp\sqrt{t}}{\delta} \right)^{n_1+n_2}.$$

We next show that

$$\mathcal{F}_3^{(n_1, n_2)}(0, 0) - (-1)^{n_1} \tilde{\mathcal{F}}_3^{(n_1, n_2)}(0, 0) = i^{n_1+n_2} \mathcal{F}_1^{(n_1, n_2)}(0, 0). \quad (7.9)$$

For this, we compute

$$\mathcal{F}_3^{(n_1, n_2)}(0, 0) - (-1)^{n_1} \tilde{\mathcal{F}}_3^{(n_1, n_2)}(0, 0) = a_0^{(n_1, n_2)}(0, 0) - a_3^{(n_1, n_2)}(0, 0).$$

Since  $a_3(-x_1, -x_2) = -a_3(x_1, x_2)$ , we obtain

$$a_3^{(n_1, n_2)}(0, 0) = (-1)^{n_1+n_2+1} a_3^{(n_1, n_2)}(0, 0).$$

Since in the sums of interest  $n_1 \equiv n_2 \pmod{2}$ , the contribution of  $a_3$  vanishes. As claimed, we are left with

$$a_0^{(n_1, n_2)}(0, 0) = i^{n_1+n_2} \left[ \frac{\partial^{n_1}}{\partial x_1^{n_1}} \frac{\partial^{n_2}}{\partial x_2^{n_2}} e^{-Q(x_1, x_2)} \right]_{x_1=x_2=0} = i^{n_1+n_2} \mathcal{F}_1^{(n_1, n_2)}(0, 0).$$

Finally, the contribution from  $H_1$  gives, observing that the Euler-Maclaurin main term vanishes

$$\sum_{0 \leq r \leq \frac{kp}{\delta} - 1} e^{-2\pi i \frac{h}{k} \left( r + \frac{1}{p} \right)^2} \sum_{m \geq 0} \frac{B_{2m+1} \left( \frac{\delta \left( r + \frac{1}{p} \right)}{kp} \right)}{(2m+1)!} \mathcal{F}_4^{(2m)}(0) \left( \frac{k^2 p^2}{\delta^2} t \right)^m$$

with  $\mathcal{F}_4(x) := M(\sqrt{\frac{2}{\pi}}x)e^{x^2}$ . The claim then follows, observing that

$$\mathcal{F}_4^{(2m)}(0) = (-1)^{m+1} \left[ \frac{\partial^{2m}}{\partial x^{2m}} e^{-x^2} \right]_{x=0} = (-1)^{m+1} \mathcal{F}_2^{(2m)}(0). \quad (7.10)$$

□

**7.2. Asymptotics of  $\mathcal{E}_2$ .** Writing

$$F_2 \left( e^{2\pi i \frac{h}{k} - t} \right) \sim \sum_{m \geq 0} b_{h,k}(m) t^m \quad (t \rightarrow 0^+),$$

we next show

**Theorem 7.2.** *We have*

$$\mathbb{E}_2 \left( \frac{h}{k} + \frac{it}{2\pi} \right) \sim \sum_{m \geq 0} b_{-h,k}(m) (-t)^m \quad (t \rightarrow 0^+).$$

*Proof.* We write, using Lemma 6.2 and (6.3)

$$\mathbb{E}_2(\tau) = \mathcal{E}_{2,1}(\tau) + \mathcal{E}_{2,2}(\tau),$$

where

$$\begin{aligned} \mathcal{E}_{2,1}(\tau) &:= \frac{1}{2} \sum_{\alpha \in \mathcal{S}} \eta(\alpha) \sum_{n \in \alpha + \mathbb{N}_0^2} n_2 M_2 \left( \sqrt{3}; \sqrt{3v}(2n_1 + n_2), \sqrt{vn_2} \right) q^{-Q(n_1, n_2)} \\ &\quad + \frac{1}{2} \sum_{\alpha \in \tilde{\mathcal{S}}} \tilde{\eta}(\alpha) \sum_{n \in \alpha + \mathbb{N}_0^2} n_2 M_2 \left( \sqrt{3}; \sqrt{3v}(-2n_1 + n_2), \sqrt{vn_2} \right) q^{-Q(-n_1, n_2)}, \\ \mathcal{E}_{2,2}(\tau) &:= \frac{1}{4\pi\sqrt{v}} \sum_{\alpha \in \mathcal{S}} \eta(\alpha) \sum_{n \in \alpha + \mathbb{N}_0^2} e^{-\pi(3n_1 + 2n_2)^2 v} M \left( \sqrt{3v}n_1 \right) q^{-Q(n_1, n_2)} \\ &\quad + \frac{1}{4\pi\sqrt{v}} \sum_{\alpha \in \tilde{\mathcal{S}}} \tilde{\eta}(\alpha) \sum_{n \in \alpha + \mathbb{N}_0^2} e^{-\pi(-3n_1 + 2n_2)^2 v} M \left( \sqrt{3v}n_1 \right) q^{-Q(-n_1, n_2)}, \end{aligned}$$

where  $\tilde{\eta}(\alpha) := \eta(1 - \alpha_1, \alpha_2)$ . We then again use (7.2) to split

$$\mathcal{E}_{2,1}(\tau) = \mathcal{E}_2^*(\tau) + H_2(\tau),$$

where

$$\begin{aligned} \mathcal{E}_2^*(\tau) &:= \frac{1}{2} \sum_{\alpha \in \mathcal{S}} \eta(\alpha) \sum_{n \in \alpha + \mathbb{N}_0^2} n_2 M_2^* \left( \sqrt{3}; \sqrt{3v}(2n_1 + n_2), \sqrt{vn_2} \right) q^{-Q(n_1, n_2)} \\ &\quad + \frac{1}{2} \sum_{\alpha \in \tilde{\mathcal{S}}} \tilde{\eta}(\alpha) \sum_{n \in \alpha + \mathbb{N}_0^2} n_2 M_2^* \left( \sqrt{3}; \sqrt{3v}(-2n_1 + n_2), \sqrt{vn_2} \right) q^{-Q(-n_1, n_2)}, \\ H_2(\tau) &:= \frac{1}{2} \sum_{\beta \in \left\{ \frac{1}{p}, 1 - \frac{1}{p} \right\}} \sum_{m \in \beta + \mathbb{N}_0} m M(2\sqrt{v}m) q^{-m^2}. \end{aligned}$$

We first investigate  $\mathcal{E}_2^*$ . We write, with  $\mathcal{G}_3(x_1, x_2) := x_2 \mathcal{F}_3(x_1, x_2)$  and  $\tilde{\mathcal{G}}_3(x_1, x_2) := \mathcal{G}_3(-x_1, x_2)$ ,

$$\begin{aligned} \mathcal{E}_2^* \left( \frac{h}{k} + it \right) &= \frac{1}{\sqrt{t}} \sum_{\alpha \in \mathcal{S}} \eta(\alpha) \sum_{0 \leq \ell \leq kp-1} e^{-2\pi i \frac{h}{k} Q(\ell_1 + \alpha_1, \ell_2 + \alpha_2)} \sum_{n \in \frac{\ell + \alpha}{kp} + \mathbb{N}_0^2} \mathcal{G}_3(kp\sqrt{t}n) \\ &\quad + \frac{1}{\sqrt{t}} \sum_{\alpha \in \tilde{\mathcal{S}}} \tilde{\eta}(\alpha) \sum_{0 \leq \ell \leq kp-1} e^{-2\pi i \frac{h}{k} Q(-\ell_1 - \alpha_1, \ell_2 + \alpha_2)} \sum_{n \in \frac{\ell + \alpha}{kp} + \mathbb{N}_0^2} \tilde{\mathcal{G}}_3(kp\sqrt{t}n). \end{aligned}$$

The contribution from  $\mathcal{G}_3$  to the main term is, as in Subsection 4.2

$$\frac{1}{k^2 p^2 t^{\frac{3}{2}}} \mathcal{I}_{\mathcal{G}_3} \sum_{\alpha \in \mathcal{S}} \eta(\alpha) \sum_{0 \leq \ell \leq kp-1} e^{-2\pi i \frac{h}{k} Q(\ell + \alpha)} = 0.$$

In the same way we see that the contribution from  $\tilde{\mathcal{G}}_3$  to the main term vanishes.

The contribution from  $\mathcal{G}_3$  to the second term in Euler-Maclaurin is, as in Subsection 4.2,

$$\begin{aligned} -2 \sum_{\alpha \in \mathcal{S}^*} \sum_{0 \leq \ell \leq kp-1} e^{-2\pi i \frac{h}{k} Q(\ell + \alpha)} \sum_{n_2 \geq 0} \frac{B_{2n_2+1} \left( \frac{\ell_2 + \alpha_2}{kp} \right)}{(2n_2 + 1)!} \\ \times \int_0^\infty \left( \mathcal{G}_3^{(0, 2n_2)}(x_1, 0) + \tilde{\mathcal{G}}_3^{(0, 2n_2)}(x_1, 0) \right) dx_1 (kp)^{2n_2-1} t^{n_2-1}. \end{aligned}$$

We claim that

$$\int_0^\infty \left( \mathcal{G}_3^{(0, 2n_2)}(x_1, 0) + \tilde{\mathcal{G}}_3^{(0, 2n_2)}(x_1, 0) \right) dx_1 = (-1)^{n_2+1} \int_0^\infty \mathcal{G}_1^{(0, 2n_2)}(x_1, 0) dx_1. \quad (7.11)$$

Since we need to differentiate the  $x_2$ -factor exactly once, we have

$$\begin{aligned} \mathcal{G}_3^{(0, 2n_2)}(x_1, 0) + \tilde{\mathcal{G}}_3^{(0, 2n_2)}(x_1, 0) &= 2n_2 \left( \mathcal{F}_3^{(0, 2n_2-1)}(x_1, 0) + \tilde{\mathcal{F}}_3^{(0, 2n_2-1)}(x_1, 0) \right), \\ \mathcal{G}_1^{(0, 2n_2)}(x_1, 0) &= 2n_2 \mathcal{F}_1^{(0, 2n_2-1)}(x_1, 0). \end{aligned}$$

The claim (7.11) then follows from (7.3).

The third term in Euler-Maclaurin vanishes as in Subsection 4.2.

For the final term in the Euler-Maclaurin summation formula, we obtain

$$\begin{aligned} 2 \sum_{\alpha \in \mathcal{S}^*} \sum_{0 \leq \ell \leq kp-1} e^{-2\pi i \frac{h}{k} Q(\ell + \alpha)} \sum_{\substack{n_1, n_2 \geq 0 \\ n_1 \not\equiv n_2 \pmod{2}}} \frac{B_{n_1+1} \left( \frac{\ell_1 + \alpha_1}{kp} \right)}{(n_1 + 1)!} \frac{B_{n_2+1} \left( \frac{\ell_2 + \alpha_2}{kp} \right)}{(n_2 + 1)!} \\ \times \left( \mathcal{G}_3^{(n_1, n_2)}(0, 0) + (-1)^{n_1+1} \tilde{\mathcal{G}}_3^{(n_1, n_2)}(0, 0) \right) (kp)^{n_1+n_2} t^{\frac{n_1+n_2-1}{2}}. \end{aligned}$$

Using (7.9) the claim follows, computing

$$\begin{aligned} \mathcal{G}_3^{(n_1, n_2)}(0, 0) + (-1)^{n_1+1} \tilde{\mathcal{G}}_3^{(n_1, n_2)}(0, 0) &= n_2 \left( \mathcal{F}_3^{(n_1, n_2-1)}(0, 0) - (-1)^{n_1} \tilde{\mathcal{F}}_3^{(n_1, n_2-1)}(0, 0) \right) \\ &= n_2 i^{n_1+n_2-1} \mathcal{F}_1^{(n_1, n_2-1)}(0, 0) = i^{n_1+n_2-1} \mathcal{G}_1^{(n_1, n_2)}(0, 0). \end{aligned}$$

We next consider  $H_2$ . We have, with  $\mathcal{G}_4(x) := x\mathcal{F}_4(x)$ ,

$$H_2\left(\frac{h}{k} + \frac{it}{2\pi}\right) = \frac{1}{2\sqrt{t}} \sum_{\beta \in \left\{\frac{1}{p}, 1-\frac{1}{p}\right\}} \sum_{0 \leq r \leq kp-1} e^{-2\pi i \frac{h}{k}(r+\beta)^2} \sum_{m \in \frac{r+\beta}{kp} + \mathbb{N}_0} \mathcal{G}_4(kp\sqrt{t}m).$$

The Euler-Maclaurin main term again vanishes, using (4.3). The second term is

$$- \sum_{0 \leq r \leq kp-1} e^{2\pi i \frac{h}{k}\left(r+\frac{1}{p}\right)^2} \sum_{m \geq 0} \frac{B_{2m+2}\left(\frac{r+\frac{1}{p}}{kp}\right)}{(2m+2)!} \mathcal{G}_4^{(2m+1)}(0) (kp)^{2m+1} t^m,$$

pairing again  $\beta$  and  $1-\beta$ . Using (7.10) we then obtain

$$\mathcal{G}_4^{(2m+1)}(0) = (2m+1)\mathcal{F}_4^{(2m)}(0) = (2m+1)(-1)^{m+1}\mathcal{F}_2^{(2m)}(0) = (-1)^{m+1}\mathcal{G}_2^{(2m+1)}(0).$$

Finally, we show that  $\mathcal{E}_{2,2}$  is asymptotically 0. Using that  $\lim_{x \rightarrow 0^+} M^*(\pm x) = \mp 1$ , we split

$$\mathcal{E}_{2,2}(\tau) = \mathcal{E}_{2,2}^*(\tau) + H_3(\tau),$$

where

$$\begin{aligned} \mathcal{E}_{2,2}^*(\tau) &:= \frac{1}{4\pi\sqrt{v}} \sum_{\alpha \in \mathcal{S}} \eta(\alpha) \sum_{n \in \alpha \in \mathbb{N}_0^2} e^{-\pi(3n_1+2n_2)^2v} M^*\left(\sqrt{3v}n_1\right) q^{-Q(n_1,n_2)} \\ &\quad + \frac{1}{4\pi\sqrt{v}} \sum_{\alpha \in \tilde{\mathcal{S}}} \tilde{\eta}(\alpha) \sum_{n \in \alpha \in \mathbb{N}_0^2} e^{-\pi(-3n_1+2n_2)^2v} M^*\left(-\sqrt{3v}n_1\right) q^{-Q(-n_1,n_2)}, \\ H_3(\tau) &:= \frac{1}{4\pi\sqrt{v}} \sum_{\beta \in \left\{\frac{1}{p}, 1-\frac{1}{p}\right\}} \sum_{m \in \beta + \mathbb{N}_0} e^{-4\pi m^2v} q^{-m^2}. \end{aligned}$$

We first investigate the asymptotic behavior of  $\mathcal{E}_{2,2}$ . We write

$$\begin{aligned} \mathcal{E}_{2,2}^*\left(\frac{h}{k} + \frac{it}{2\pi}\right) &= \frac{1}{\sqrt{\pi t}} \sum_{\alpha \in \mathcal{S}} \eta(\alpha) \sum_{0 \leq \ell \leq kp-1} e^{-2\pi i \frac{h}{k}Q(\ell_1+\alpha_1, \ell_2+\alpha_2)} \sum_{n \in \frac{\ell+\alpha}{kp} + \mathbb{N}_0^2} \mathcal{G}_5(kp\sqrt{t}n) \\ &\quad + \frac{1}{\sqrt{\pi t}} \sum_{\alpha \in \tilde{\mathcal{S}}} \tilde{\eta}(\alpha) \sum_{0 \leq \ell \leq kp-1} e^{-2\pi i \frac{h}{k}Q(-\ell_1-\alpha_1, \ell_2+\alpha_2)} \sum_{n \in \frac{\ell+\alpha}{kp} + \mathbb{N}_0^2} \tilde{\mathcal{G}}_5(kp\sqrt{t}n), \end{aligned}$$

where

$$\mathcal{G}_5(x_1, x_2) := \frac{1}{2\sqrt{2}} e^{-\frac{3x_1^2}{2} - 3x_1x_2 - x_2^2} M^*\left(\sqrt{\frac{3}{2\pi}}x_1\right), \quad \tilde{\mathcal{G}}_5(x_1, x_2) := \mathcal{G}_5(-x_1, x_2).$$

As before the main term and the third term in the Euler-Maclaurin summation formula go away. The second term equals

$$- \frac{2}{\sqrt{\pi t}} \sum_{\alpha \in \mathcal{S}^*} \sum_{0 \leq \ell \leq kp-1} e^{-2\pi i \frac{h}{k}Q(\ell+\alpha)}$$

$$\times \sum_{n_2 \geq 0} \frac{B_{2n_2+1} \left( \frac{\ell_2 + \alpha_2}{kp} \right)}{(2n_2 + 1)!} \int_0^\infty \left( \mathcal{G}_5^{(0,2n_2)}(x_1, 0) + \tilde{\mathcal{G}}_5^{(0,2n_2)}(x_1, 0) \right) dx_1 (k^2 p^2 t)^{n_2}.$$

It is however not hard to see that

$$\mathcal{G}_5^{(0,2n_2)}(x_1, 0) + \mathcal{G}_5^{(0,2n_2)}(x_1, 0) = 0.$$

The final term in the Euler-Maclaurin summation formula is

$$\begin{aligned} \frac{2}{\sqrt{\pi t}} \sum_{\alpha \in \mathcal{S}^*} \eta(\alpha) \sum_{0 \leq \ell \leq kp-1} e^{-2\pi i \frac{h}{k} Q(\ell + \alpha)} \sum_{\substack{n_1, n_2 \geq 0 \\ n_1 \not\equiv n_2 \pmod{2}}} \frac{B_{n_1+1} \left( \frac{\ell_1 + \alpha_1}{kp} \right)}{(n_1 + 1)!} \frac{B_{n_2+1} \left( \frac{\ell_2 + \alpha_2}{kp} \right)}{(n_2 + 1)!} \\ \times \left( \mathcal{G}_5^{(n_1, n_2)}(0, 0) + (-1)^{n_1+1} \tilde{\mathcal{G}}_5^{(n_1, n_2)}(0, 0) \right) (kp\sqrt{t})^{n_1+n_2}. \end{aligned}$$

Now

$$\mathcal{G}_5^{(n_1, n_2)}(0, 0) + (-1)^{n_1+1} \tilde{\mathcal{G}}_5^{(n_1, n_2)}(0, 0) = \mathcal{G}_{5,1}^{(n_1, n_2)}(0, 0), \quad (7.12)$$

where  $\mathcal{G}_{5,1}(x_1, x_2) := -e^{-\frac{3x_1^2}{2} - 3x_1x_2 - x_2^2}$ . Now since  $n_1 \not\equiv n_2 \pmod{2}$ , (7.12) vanishes.

Finally

$$H_3 \left( \frac{h}{k} + \frac{it}{2\pi} \right) = \frac{1}{2\sqrt{2\pi t}} \sum_{\beta \in \left\{ \frac{1}{p}, 1 - \frac{1}{p} \right\}} \sum_{0 \leq r \leq kp-1} e^{-2\pi i \frac{h}{k} (r+\beta)^2} \sum_{m \in \frac{r+\beta}{kp} + \mathbb{N}_0} \mathcal{F}_2(kp\sqrt{t}m).$$

The Euler-Maclaurin main term is, using (4.1),

$$-\frac{1}{2kp\sqrt{2\pi t}} \mathcal{I}_{\mathcal{F}_2} \sum_{\beta \in \left\{ \frac{1}{p}, 1 - \frac{1}{p} \right\}} \sum_{r \pmod{kp}} e^{-2\pi i \frac{h}{k} (r+\beta)^2} = 0.$$

The remaining term is, since  $\mathcal{F}_2$  is an even function

$$\frac{1}{\sqrt{2\pi t}} \sum_{0 \leq r \leq kp-1} e^{-2\pi i \frac{h}{k} (r+\beta)^2} \sum_{m \geq 0} \frac{B_{2m+2} \left( \frac{1}{p} \right)}{(2m+2)!} \mathcal{F}_2^{(2m+1)}(0) = 0.$$

**7.3. Proof of Theorem 1.1.** We are now ready to prove a refined version of Theorem 1.1.

**Theorem 7.3.** (1) The function  $\hat{F}_1 : \mathbb{Q} \rightarrow \mathbb{C}$  defined by  $\hat{F}_1\left(\frac{h}{k}\right) := F_1(e^{2\pi i \frac{ph}{k}})$  is a depth two quantum modular form of weight one for  $\Gamma_p$  with multiplier  $\left(\frac{-3}{d}\right)$ .

(2) The function  $\hat{F}_2 : \mathcal{Q}_1 \rightarrow \mathbb{C}$  defined by  $\hat{F}_2\left(\frac{h}{k}\right) := F_2(e^{2\pi i \frac{ph}{k}})$  is a depth two quantum modular form of weight two for  $\Gamma_p$  with multiplier  $\left(\frac{3}{d}\right)$  and quantum set

$$\mathcal{Q}_1 := \left\{ \frac{h}{k} \in \mathbb{Q} : \gcd(h, k) = 1, p|k \right\} \subset \mathbb{Q}.$$

*Proof.* (1) We have, by Theorem 7.1,

$$\hat{F}_1 \left( \frac{h}{k} \right) = \lim_{t \rightarrow 0^+} F_1 \left( e^{2\pi i \frac{ph}{k} - t} \right) = a_{hp_1, \frac{k}{p_2}}(0) = \lim_{t \rightarrow 0^+} \mathbb{E}_1 \left( -\frac{h}{k} + \frac{it}{2\pi} \right),$$

where  $p_1 := p/\gcd(k, p)$ ,  $p_2 := \gcd(k, p)$ . Proposition 5.2 then gives the claim.

(2) Theorem 7.1 gives, for  $p|k$

$$\widehat{F}_2\left(\frac{h}{k}\right) = \lim_{t \rightarrow 0^+} \widehat{F}_2\left(e^{2\pi i \frac{ph}{k} - t}\right) = a_{h, \frac{k}{p}}(0) = \lim_{t \rightarrow 0^+} \mathbb{E}_2\left(-\frac{h}{k} + \frac{it}{2\pi}\right).$$

Proposition 5.3 then gives the claim.  $\square$

*Remark.* For odd  $d$ , we have that  $(\frac{3}{d}) = (\frac{-3}{d}) = 1$  if and only if  $d \equiv 1 \pmod{12}$  so that both  $F_1$  and  $F_2$  can be viewed as quantum modular forms with the trivial character under a suitable subgroup of  $\Gamma_p$  (e.g. the principal congruence subgroup  $\Gamma(24p)$ ).  $\square$

## 8. COMPLETED INDEFINITE THETA FUNCTIONS

In this section, we embed the double Eichler integrals in a modular context by viewing them as “purely non-holomorphic” parts of indefinite theta series.

**8.1. Weight one.** The functions  $E_2$  and  $M_2$  were introduced in [1], where they played a crucial role in understanding modular indefinite theta functions of signature  $(j, 2)$  ( $j \in \mathbb{N}_0$ ). We consider the quadratic form  $Q_1(n) := \frac{1}{2}n^T A_1 n$  and the bilinear form  $B_1(n, m) := n^T A_1 m$  given by  $A_1 := \begin{pmatrix} 6 & 3 & 6 & 3 \\ 3 & 2 & 3 & 2 \\ 6 & 3 & 0 & 0 \\ 3 & 2 & 0 & 0 \end{pmatrix}$ , and define  $A_0 := \begin{pmatrix} 6 & 3 \\ 3 & 2 \end{pmatrix}$ ,  $P_0(n_1, n_2) := M_2(\sqrt{3}; \sqrt{3}(2n_1 + n_2), n_2)$  and, for  $n \in \mathbb{R}^4$ , set

$$\begin{aligned} P(n) := & M_2\left(\sqrt{3}; \sqrt{3}(2n_3 + n_4), n_4\right) + (\operatorname{sgn}(2n_3 + n_4) + \operatorname{sgn}(n_1))(\operatorname{sgn}(3n_3 + 2n_4) + \operatorname{sgn}(n_2)) \\ & + (\operatorname{sgn}(n_4) + \operatorname{sgn}(n_2))M_1\left(\sqrt{3}(2n_3 + n_4)\right) + (\operatorname{sgn}(n_3) + \operatorname{sgn}(n_1))M_1(3n_3 + 2n_4). \end{aligned}$$

Note that, for  $\alpha \in \mathcal{S}^*$ ,

$$2\mathcal{E}_{1, \alpha}(\tau) = \Theta_{-A_0, P_0, \alpha}(\tau).$$

We view this function as “purely non-holomorphic” part of the indefinite theta function

$$\Theta_{A_1, P, a}(\tau) = \sum_{n \in a + \mathbb{Z}^4} P(\sqrt{v}n) q^{Q_1(n)}, \quad (8.1)$$

where  $a \in \frac{1}{p}A_1^{-1}\mathbb{Z}^4$  with  $(a_3, a_4) = (\alpha_1, \alpha_2)$ . One can either employ Section 4.3 of [1] or proceed directly (as we do here) to prove the following proposition.

**Proposition 8.1.** *Assume that  $a \in \frac{1}{p}A_1^{-1}\mathbb{Z}^4$  with  $a_1, a_2, a_4 \notin \mathbb{Z}$ .*

(1) *We have*

$$\Theta_{A_1, P^-, a}(\tau) = 2\mathcal{E}_{1, (a_3, a_4)}(\tau)\Theta_{A_0, 1, (a_1 - a_3, a_2 - a_4)}(\tau),$$

where

$$P^-(n_1, n_2, n_3, n_4) := M_2\left(\sqrt{3}; \sqrt{3}(2n_3 + n_4), n_4\right).$$

(2) *The functions  $\Theta_{A_1, P, a}$  and  $\Theta_{A_0, P_0, (a_3, a_4)}$  converge absolutely and locally uniformly.*

(3) *The function  $\tau \mapsto \Theta_{A_1, P, a}(p\tau)$  transforms like a modular form of weight two for some subgroup of  $\mathrm{SL}_2(\mathbb{Z})$  and some character.*

*Remark.* When considering indefinite theta functions of signature  $(j, 2)$ , one usually obtains four  $M_2$ -terms as the purely non-holomorphic" part. The arguments of these four  $M_2$ -functions are dictated by the holomorphic part. The fact that  $(1, 0, 0, 0)^T$  and  $(0, 1, 0, 0)^T$  (which correspond to  $n_1$  and  $n_2$  occuring in  $P$ ) have norm zero with respect to  $A_1^{-1}$  causes the "missing"  $M_2$ -terms to vanish. Therefore we refer to this situation as a *double null limit* (see [1]).

*Proof of Proposition 8.1.* (1) Shifting  $(n_1, n_2, n_3, n_4) \mapsto (n_1 - n_3, n_2 - n_4, n_3, n_4)$  on the left hand side of the identity gives the claim.

(2) For  $\Theta_{-A_0, P_0, (a_3, a_4)}$  we employ the asymptotic given in (2.7), to obtain

$$\begin{aligned} \left| M_2 \left( \sqrt{3}; \sqrt{3v} (2n_1 + n_2), \sqrt{vn_2} \right) q^{\frac{1}{2}n^T A_0 n} \right| &\leq \frac{e^{-\pi(3(2n_1+n_2)^2+n_2^2)v}}{\pi^2 n_1 n_2} e^{\pi n^T A_0 n v} \\ &\leq c_1 e^{-2\pi n^T A_0 n v} e^{\pi n^T A_0 n v} = c_1 e^{-\pi n^T A_0 n v} \end{aligned}$$

for some  $c_1 \in \mathbb{R}^+$  and  $(n_1, n_2) \in (a_3, a_4) + \mathbb{Z}^2$  with  $n_1 \neq 0$ . By plugging in the definition, one can show that for some  $c_2 \in \mathbb{R}^+$  and  $n = (0, n_2) \in (a_3, a_4) + \mathbb{Z}^2$

$$\left| M_2 \left( \sqrt{3}; \sqrt{3vn_2}, \sqrt{vn_2} \right) q^{\frac{1}{2}n^T A_0 n} \right| \leq c_2 e^{-\pi n^T A_0 n v}.$$

Using that  $A_0$  is positive definite, we obtain, for some  $c_3 \in \mathbb{R}^+$

$$\sum_{n \in (a_3, a_4) + \mathbb{Z}^2} \left| M_2 \left( \sqrt{3}; \sqrt{3v} (2n_1 + n_2), \sqrt{vn_2} \right) q^{\frac{1}{2}n^T A_0 n} \right| \leq c_3 \sum_{n \in (a_3, a_4) + \mathbb{Z}^2} e^{-\pi n^T A_0 n v} < \infty,$$

implying the absolute and locally uniform convergence of  $\Theta_{-A_0, P_0, (a_3, a_4)}$ . Combining this with (1) and the convergence of the positive definite theta series  $\Theta_{A_0, 1, (a_1 - a_3, a_2 - a_4)}$ , we obtain absolute and locally uniform convergence of the  $M_2$ -part of  $\Theta_{A_1, P, a}$ .

For the part containing only sign-terms

$$\sum_{n \in a + \mathbb{Z}^4} (\text{sgn}(2n_3 + n_4) + \text{sgn}(n_1)) (\text{sgn}(3n_3 + 2n_4) + \text{sgn}(n_2)) q^{Q_1(n)}, \quad (8.2)$$

we consider the determinant of  $\Delta_{A_1}(n, b_1, b_2, b_3, b_4)$ , where  $(\Delta_M(v_1, \dots, v_5))_{j, \ell} := v_j^T M v_\ell$  and

$$(b_1, b_2, b_3, b_4) := \frac{1}{3} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 3 \\ 2 & -3 & -1 & 0 \\ -3 & 6 & 0 & -3 \end{pmatrix}.$$

We compute the determinant  $\det(\Delta_{A_1}(n, b_1, b_2, b_3, b_4))$  via Laplace expansion to obtain

$$\begin{aligned} e^{-\pi v Q_1(n)} &\leq e^{-\pi \left( \frac{15}{16} B_1(b_1, n)^2 + \frac{2}{9} B_1(b_2, n)^2 + B_1(b_1, n) B_1(b_3, n) + 2 B_1(b_2, n) B_1(b_4, n) \right) v} \\ &\leq e^{-c_4 (B_1(b_1, n)^2 + B_1(b_2, n)^2 + |B_1(b_3, n)| + |B_1(b_4, n)|) v} \end{aligned}$$

with some  $c_4 \in \mathbb{R}^+$  for all  $n \in a + \mathbb{Z}^4$  which satisfy the condition

$$(\text{sgn}(2n_3 + n_4) + \text{sgn}(n_1)) (\text{sgn}(3n_3 + 2n_4) + \text{sgn}(n_2)) \neq 0.$$

Thus (8.2) is dominated by

$$\sum_{n \in a + \mathbb{Z}^4} \left| (\text{sgn}(2n_3 + n_4) + \text{sgn}(n_1)) (\text{sgn}(3n_3 + 2n_4) + \text{sgn}(n_2)) e^{-\pi Q_1(n)v} \right|$$

$$\leq 4 \sum_{n \in a + \mathbb{Z}^4} e^{-c_4(B_1(b_1, n)^2 + B_1(b_2, n)^2 + |B_1(b_3, n)| + |B_1(b_4, n)|)} v < \infty.$$

To deal with the contribution of the third and fourth summand of  $P$  one combines the approaches of the two previous terms.

(3) We use Lemma 2.1 to rewrite  $P$  as a limit of  $E_2$ -functions, namely

$$P(n) = \lim_{\varepsilon \rightarrow 0} \widehat{P}_\varepsilon(n),$$

where

$$\begin{aligned} \widehat{P}_\varepsilon(n) := & \left( E_2 \left( \frac{\varepsilon}{3}; \sqrt{3}(2n_3 + n_4), -\varepsilon \left( n_1 + n_3 + \frac{n_4}{\sqrt{3}} \right) + \frac{3n_2}{\varepsilon(2\sqrt{3} - 3)} \right) \right. \\ & + E_2 \left( \frac{\varepsilon}{2}; (3n_3 + 2n_4), \frac{3n_1}{\varepsilon(2\sqrt{3} - 3)} - \varepsilon(n_2 + \sqrt{3}n_3 + n_4) \right) + E_2 \left( \sqrt{3}; \sqrt{3}(2n_3 + n_4), n_4 \right) \\ & \left. + E_2 \left( -\sqrt{3}; \frac{n_2}{2\varepsilon} - \frac{\varepsilon}{2}(n_2 + 2n_4), \frac{\sqrt{3}}{2\varepsilon}(2n_1 + n_2) - \frac{\sqrt{3}}{2}\varepsilon(2n_1 + n_2 + 4n_3 + 2n_4) \right) \right). \end{aligned}$$

One can then verify that each occurring term  $E_2(\kappa; b^T n, c^T n)$  satisfies the Vignéras differential equation given in Theorem 2.2 with  $\lambda = 0$  and  $A = A_1$ . A straightforward calculation shows that the Vignéras differential equation is satisfied for  $\widehat{P}_\varepsilon$  with respect to  $A_1$  if and only if it is satisfied for  $\widehat{P}_{\varepsilon, p}(n) := \widehat{P}_\varepsilon(\sqrt{p}n)$  with respect to  $pA_1$ . Furthermore, we have

$$\Theta_{A_1, P, a}(p\tau) = \Theta_{pA_1, P_p, a}(\tau) = \lim_{\varepsilon \rightarrow 0} \Theta_{pA_1, \widehat{P}_{\varepsilon, p}, a}(\tau)$$

where  $P_p(n) := P(\sqrt{p}n)$ . We can apply Theorem 2.2 to obtain weight 2 modularity of  $\Theta_{pA_1, \widehat{P}_{\varepsilon, p}, a}$  since  $a \in (pA_1)^{-1}\mathbb{Z}^4$ . Now, taking the limit  $\varepsilon \rightarrow 0$  proves the claim.  $\square$

**8.2. Completion: weight two.** Similarly as in the previous Section 8.1, the function  $\mathbb{E}_2$  may be related to a modular object of weight three. This connection becomes evident when writing  $\mathbb{E}_2$  as a Jacobi derivative as in Theorem 6.2. We leave the details to the reader.

**8.3. Lowering.** The indefinite theta series considered in Subsection 8.1 are higher depth harmonic Maass forms following Zagier-Zwegers. Roughly speaking, by this we mean that applying the *Maass lowering operator*  $L := -2iv^2 \frac{\partial}{\partial \bar{\tau}}$  makes the function simpler. In particular, for the iterated Eichler integral, we have

$$L(I_{f_1, f_2}(\tau)) = 2^{k_1} v^{k_1} f_1(-\bar{\tau}) I_{f_2}(\tau).$$

Now  $v^{k_1} f_1(-\bar{\tau})$  is  $v^{k_1}$  times a conjugated modular form of weight  $k_1$  (so transforming of weight  $-k_1$ ) and  $I_{f_2}$ , defined in (2.11), is the non-holomorphic part of an harmonic Maass form of weight  $2 - k_2$ .



## 9. CONCLUSION AND FURTHER QUESTIONS

We conclude here we several comments and research directions

- (1) We plan more systematically study higher depth quantum modular forms and to describe explicitly the quantum  $S$ -modular matrix of  $F(q)$ , including the  $p = 2$  case. This requires modification of several arguments used here for  $F_2(q)$  (note that we restricted ourselves to  $\Gamma_p$  out of necessity). This result would allow us to make a more precise connection between  $W(p)_{A_2}$  and its irreducible modules. For one, we should be able to associate an  $S$ -matrix to the set of atypical irreducible  $W(p)_{A_2}$ -characters, in parallel to [7]. Our conjecture is that the quantum  $S$ -matrix is equivalent to the  $S$ -matrix of level  $p - 3$  Wess-Zumino-Witten model of  $sl_3$ . For the Lie algebra of type  $sl_2$ , a similar result (also based on quantum modularity) was proven in [8].
- (2) Iterated (or multiple) Eichler integrals studied in Section 5 are of independent interest. As in other theories dealing with iterated integrals (e.g. non-commutative modular symbols, Chen's integrals and multiple zeta-values) shuffle relations are expected to play an important role. Another goal worth pursuing is to connect iterated Eichler integrals of half-integral weights to Manin's work [18].
- (3) We plan to investigate the asymptotic of  $F(q)$  in terms of finite  $q$ -series evaluated at root of unity. This requires certain hypergeometric type formulas for double rank two false theta functions.
- (4) Let  $f \in S_{\frac{3}{2}}(\chi_1, \Gamma) \otimes S_{\frac{3}{2}}(\chi_2, \Gamma)$  with  $\chi_j$  multipliers and  $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ . We plan to connect the error of modularity

$$\int_0^{i\infty} \int_0^{w_1} \frac{f(w_1, w_2)}{\sqrt{-i(w_1 + \tau)} \sqrt{-i(w_2 + \tau)}} dw_2 dw_1$$

of suitable  $f$  to a "double Mordell" integral. In the rank one this connection is well-understood [27, Theorem 1.16].

- (5) Very recently, W. Yuasa [23] gave an explicit formula for the *tail* of  $(2, 2p)$ -torus links associated to the sequence of colored Jones polynomials:  $J_{n\omega_j}(K, q)$ ,  $n \in \mathbb{N}$ , where  $\omega_j$ ,  $j = 1, 2$  are the fundamental weights. We were able to identify the same tail as a summand of  $F(q)$ , up to the factor  $1 - q$  (viz. extract the "diagonal"  $m_1 = m_2$  in formula (1.7)). This raises the following question: Is it true that  $F(q)$  is the tail of  $J_{n\rho}(K, q)$ , ( $n \in \mathbb{N}$ ) (here  $\rho = \omega_1 + \omega_2$ ), up to a rational function of  $q$ ? For related computations of tails colored with  $\mathfrak{sl}_3$  representations see [12].

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